Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VI, Częstochowa 1999

On a sufficirnt condition for the univalency of holomorphic functions

Jaroslav Fuka

Let us denote by C the complex plane, by $D_r(z)$ the open disc with the center z and the radius r and by $T_r(z)$ its boundary circle. For $z = 0$ (and $r = 1$) we simply write D_r, T_r (and D, T). Let $Q \subset C$ be a compact, z a given point in C. Let $N_z^Q(\theta)$ be the total number of points lying in the intersection of Q with the open ray $\{z + te^{i\theta};\ t > 0\}$. It is well known that this function is Borel measurable and we may adopt the following

Definition. If $Q \subset C$ is compact, we define for any $z \in C$

$$v^Q(z) = \int_0^{2\pi} N_z^Q(\theta)d\theta.$$

Further we put

$$V(Q) = \sup_{\zeta \in Q} v^Q(\zeta).$$

We shall give a sketch of the proof of the following

Theorem. *Let $f(z)$ be a function holomorphic in D and let $f'(z) \neq 0$ in D. Let*

$$\sup_{0<r<1} V(f(T_r)) \leq 3\pi.$$

Then $f(z)$ is univalent in D (i.e. for $z_1 \in \mathbf{D}$, $z_2 \in \mathbf{D}$, $z_1 \neq z_2$, we have $f(z_1) \neq f(z_2)$).

Proof. Suppose that $f(z)$ is not univalent in D. We shall prove $\sup_{0<r<1} V(f(T_r)) > 3\pi$. Denote $r_0 = \sup\{r : f \text{ is univalent in } D_r\}$. Since $f'(0) \neq 0$, f is univalent in a neighborhood of the point $z = 0$ and therefore $r_0 > 0$. Since f is not univalent in D, we have $r_0 < 1$. Clearly f is univalent in D_{r_0} but f cannot be univalent on T_{r_0}, since then it would be univalent in some D_r, $r > r_0$, with regard to $f'(z) \neq 0$ in D. Therefore there exist two different points $z_1, z_2 \in T_{r_0}$, for which $f(z_1) = f(z_2) = \zeta_0$. Since $f'(z_i) \neq 0$ for $i = 1, 2$, f is homeomorphic in some neighborhood of each point z_1, z_2 and therefore there exists $\rho > 0$ so that the disc $D_\rho(\zeta_0)$ is homeomorphic to the neighborhoods U_1, U_2 of the points z_1, z_2, respectively. Now let us denote, for $r \geq r_0$, $k = 1, 2$, $C_{r,k} = T_r \cap \overline{U}_k$; $\gamma_{r,k} = f(C_{r,k})$; $u_{r,k}, v_{r,k}$ the endpoints of the arc $\gamma_{r,k}$; $\delta_{r,k}$ that arc on the circle $T_\rho(\zeta_0)$ with endpoints $u_{r,k}$, $v_{r,k}$, which has a nonvoid intersection with $f(D_{r_0})$; $V_{r,k}$ $(r > r_0)$ the interior of the Jordan curve $\gamma_{r,k} \cup \delta_{r,k}$. Finaly we denote $\gamma = f(\langle z_1, z_2 \rangle)$. Evidently γ is a Jordan curve. The analytic Jordan arcs $\gamma_{r_0,k}$, $k = 1, 2$, intersect only in the point ζ_0, where they have a common tangent; otherwise f could not be univalent in D_{r_0}. The point ζ_0 is for $r > r_0$ the interior point of $V_{r,k}$ and since $u_{r,k} \to u_{r_0,k}$, $v_{r,k} \to v_{r_0,k}$ for $r \to r_0$, $k = 1, 2$, we claim, that there exists $r^* > r_0$ so that $\gamma_{r^*,1}$, $\gamma_{r^*,2}$ intersect in the points $\zeta_{r^*,1}$, $\zeta_{r^*,2}$, one of them, say $\zeta_{r^*,1}$, lying inside of γ, and the second one, $\zeta_{r^*,2}$, outside of γ. The formal proof of this intuitively clear claim is elementary, although somewhat troublesome (it is based on the well known fact of plane topology, that two Jordan arcs lying in the quadrat, one of them connecting the left lower corner with the right upper corner and the second one connecting the two remaining corners of the quadrat, must necessarily intersect) and so we shall omit it. Now we are in a position to supply the proof of the theorem. Take a $\rho^* > 0$ so small, that the closed disc $\overline{D_{\rho^*}(\zeta_0)}$ lies in the interior of the Jordan domain bounded by the subarcs of $\gamma_{r^*,1}$, $\gamma_{r^*,2}$ with the endpoints $\zeta_{r^*,1}$, $\zeta_{r^*,2}$. Let $\zeta_1 \in f(T_{r^*})$ be the point in the interior of γ with the property $|\zeta_1 - \zeta_{r^*,1}| \geq |\zeta - \zeta_{r^*,1}|$ for $\zeta \in f(T_{r^*}) \cap \mathrm{Int}\gamma$. Choosing r^* sufficiently small we may suppose also $|\zeta_1 - \zeta_{r^*,1}| > |\zeta_0 - \zeta_{r^*,1}|$. In the point ζ_1 the analytic arc $f(T_{r^*})$ has a tangent t. Since $v^Q(z)$ does not depend on the coordinate system in C, we may take the point ζ_1 for the pole and the straight-line t for the axis of polar coordinates in C. Lead from the point ζ_1 tangents to the circle $T_{\rho^*}(\zeta_0)$ and let $\alpha > 0$ be their angle. For all the rays issuing from the point ζ_1 to the half-plane R_+ determined by the straight-line t and the point $\zeta_{r^*,1}$ we have $N_{\zeta_{r^*,1}}^{f(T_{r^*})}(\theta) \geq 2$, because such a ray has to intersect $f(T_{r^*})$ inside and outside of γ. Moreover for such rays intersecting the circle $T_{\rho^*}(\zeta_0)$ we have by the construction $N_{\zeta_{r^*,1}}^{f(T_{r^*})}(\theta) \geq 4$ (possibly with two exceptions). For the rays issuing from ζ_1 to the half-plane opposite to R_+ we have $N_{\zeta_{r^*,1}}^{f(T_{r^*})}(\theta) \geq 1$. So together we obtain $V(f(T_{r^*})) \geq v^{f(T_{r^*})}(\zeta_{r^*,1}) = \int_0^{2\pi} N_{\zeta_{r^*,1}}^{f(T_{r^*})}(\theta) d\theta \geq \int_0^{\pi} 2 \cdot d\theta + 2\alpha + \int_\pi^{2\pi} 1 \cdot d\theta = 3\pi + 2\alpha$. Hence $\sup_{0<r<1} V(f(T_r)) \geq 3\pi + 2\alpha > 3\pi$ and the theorem is proved.

Jaroslav Fuka
Department of Mathematics
J.E.Purkyně University
České mládeže 8
400 96 Ústí nad Labem, Czech Republic