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TRIANGULAR STRUCTURES AND DUALITY *

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Abstract. We introduce and study the category AFD the objects of which are generalized convergence D-posets (with more than just one greatest element) of maps into a triangle object T and the morphisms of which are sequentially continuous D-homomorphisms. The category AFD can serve as a base category for antagonistic fuzzy probability theory. AFD-measurable maps can be considered as generalized random variables and ADF-morphisms, as their dual maps, can be considered as generalized observables.

1. Introduction

In generalized probability theory (cf. [6,11,13,19,20]) basic notions are events, states (generalized probability measures), and observables (notions dual to generalized random variables). Difference posets (abbr. to D-posets) of fuzzy sets have been introduced by F. Kôpka in 1992 (see [15]). More general D-posets (cf. [16]) form a category in which classical, fuzzy, and quantum phenomena can be modeled. D-posets are equivalent to effect algebras (cf. [7]). Recall that ID (cf. [10,18]) is the category the objects of which are suitable convergence D-posets of maps into the closed unit interval $I = [0, 1]$ and the

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morphisms of which are sequentially continuous D-homomorphisms. Note that ID is a suitable base category for generalized probability theory and (unlike in the classical Kolmogorov probability theory) in ID both observables and states are morphisms.

The category theory (cf. [14]) and in particular the language of mathematical structures (cf. [1]) is natural and suitable to carry out and describe various constructions used in the foundations of generalized probability, e.g. the duality between observables and random variables. The sequential convergence and sequential continuity of the morphisms play a key role (cf. [9]).

We are motivated by the intuitionistic fuzzy probability theory (see [2,21]). Our goal is to generalize the theory of measurable spaces and measurable maps developed in [18] replacing the cogenerator $[0,1]$ by a suitable triangular object T, to prove a duality theorem, and to indicate some applications to generalized probability theory.

Recall that in intuitionistic logic the law of excluded middle does not hold. An *intuitionistic fuzzy event* $A \subseteq X$ is a pair (μ_A, ν_A) of membership functions $\mu_A, \nu_A \in I^X$ such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The intuitionistic fuzzy events are partially ordered ($(\mu_B, \nu_B) \leq (\mu_A, \nu_A)$ whenever $\mu_B \leq \mu_A$ and $\nu_A \leq \nu_B$) and carry suitable operations. *Intuitionistic fuzzy probability* sends intuitionistic fuzzy events to closed subintervals of I.

2. D-posets of fuzzy sets

Let X be a set, let $\mathcal{X} \subseteq I^X$ be a family of functions of X into I, for each $c \in I$ let $c_\mathcal{X}$ be the corresponding constant function, let "$\leq$" be the pointwise partial order on $\mathcal{X}$, and let "$\ominus$" be the pointwise partial difference defined for $v \leq u$ by $(u \ominus v)(x) = u(x) - v(x)$, $x \in X$. The quintuple $(\mathcal{X}, \leq, 0_\mathcal{X}, 1_\mathcal{X}, \ominus)$, abbreviated to $\mathcal{X}$, is a *D-poset of fuzzy sets* called an *ID-poset*. The pair $(X, \mathcal{X})$ is called an *ID-measurable space*. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be ID measurable spaces and let $f : X \longrightarrow Y$ be a map such that $v \circ f \in \mathcal{X}$ for each $v \in \mathcal{Y}$. Then f is said to be a $(\mathcal{Y}, \mathcal{X})$*-measurable map*. Let h be a map of an ID-poset $\mathcal{Y}$ into an ID-poset $\mathcal{X}$ which preserves "the ID-structure". Then h is said to be a *D-homomorphism*.

Denote MID is the category of ID-measurable spaces and measurable maps. It is known that the category ID and a distinguished subcategory of MID (consisting of sober objects) are dually naturally equivalent (cf. [10,18]).

Example 2.1. Let $(\Omega, \mathbf{A})$ be a classical measurable space. Then $\mathbf{A}$ can be considered as an ID-poset via identifying $A \in \mathbf{A}$ and its characteristic function and defining $A \ominus B = A \setminus B$ whenever $B \subseteq A$.

Example 2.2. Let $(\Omega, \mathbf{A})$ be a classical measurable space. Let $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on $\mathbf{A}$. Let $\{a\}$ be a singleton. Then each $p \in \mathcal{P}(\mathbf{A})$

is an ID-morphism of $\mathbf{A}$ into $I = I^{\{a\}}$. Denote $ev(A) = \{p(A);\ p \in \mathcal{P}(\mathbf{A})\}$ and $ev(\mathbf{A}) = \{ev(A);\ A \in \mathbf{A}\}$. For $X = \mathcal{P}(\mathbf{A})$ and $\mathcal{X} = ev(\mathbf{A})$, $(X, \mathcal{X})$ is a typical ID-measurable space.

3. AFD-system

Denote $T = \{(a, b) \in I \times I;\ a + b \leq 1\}$. Then T carries the pointwise partial order defined by $(a, b) \preceq (c, d)$ whenever $a \leq c$ and $b \leq d$, a partial difference operation defined by $(c, d) \oslash (a, b) = (c - a, d - b)$ whenever $(a, b) \preceq (c, d)$, and the pointwise sequential convergence defined by $(a, b) = \mathrm{Lim}_{n\to\infty}(a_n, b_n)$ whenever $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Denote $(T, \preceq, \oslash, \mathrm{Lim})$ the resulting structure; it will be abbreviated to T and called the *triangle* T.

Let X be a set and let T^X be the set of all maps of X into T. If X is a singleton $\{a\}$, then $T^{\{a\}}$ will be condensed to T. Let $u \in T^X$. Then there are two maps u_l and u_r of X into I such that for each $x \in X$ we have $u(x) = (u_l(x), u_r(x))$; we shall write $u = (u_l, u_r)$. In T^X there are three distinguished constants defined as follows:

$b_{\mathcal{X}} = (0_{\mathcal{X}}, 0_{\mathcal{X}}),\ b_{\mathcal{X}}(x) = (0, 0)$ for all $x \in X$;

$l_{\mathcal{X}} = (0_{\mathcal{X}}, 1_{\mathcal{X}}),\ l_{\mathcal{X}}(x) = (0, 1)$ for all $x \in X$;

$r_{\mathcal{X}} = (1_{\mathcal{X}}, 0_{\mathcal{X}}),\ r_{\mathcal{X}}(x) = (1, 0)$ for all $x \in X$.

The system T^X carries the pointwise partial order, the pointwise partial difference (we shall use the same symbols for the pointwise partial order and the pointwise partial difference on T and T^X), and the pointwise sequential convergence induced by the triangle T.

Definition 3.1. *Let X be a set and let $\mathcal{X}$ be a set of maps of X into the triangle T such that $b_{\mathcal{X}}, l_{\mathcal{X}}, r_{\mathcal{X}} \in \mathcal{X}$ and $\mathcal{X}$ is closed with respect to the pointwise partial difference. Then $\mathcal{X}$ carrying the pointwise order, the pointwise partial difference, and the pointwise sequential convergence is said to be an* AFD-system* *and $(X, \mathcal{X})$ is said to be an* AFD-measurable space. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be AFD-measurable spaces and let $f: X \longrightarrow Y$ be a map such that $v \circ f \in \mathcal{X}$ for each $v \in \mathcal{Y}$. Then f is said to be a $(\mathcal{Y}, \mathcal{X})$*-measurable map.

In what follows, all AFD-systems will be *reduced*, i.e., for each $x,\ y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$.

Denote MAFD the category of AFD-measurable spaces and measurable maps.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be AFD-measurable spaces (remember $\mathcal{X}$ and $\mathcal{Y}$ are reduced) and let $f: X \to Y$ be a $(\mathcal{Y}, \mathcal{X})$-measurable map. Define the dual map $f^{\triangleleft}$ of $\mathcal{Y}$ into $\mathcal{X}$ as follows: $f^{\triangleleft}(v) = v \circ f,\ v \in \mathcal{Y}$.

*The notion is derived from "antagonistic".

Lemma 3.2.

(i) The dual map $f^{\triangleleft}$ is sequentially continuous and preserves the structure of AFD-systems.

(ii) Let f and g be measurable maps of X into Y. If $f \neq g$, then $f^{\triangleleft} \neq g^{\triangleleft}$.

Proof. (i) First, let $\langle v_n \rangle$ be a sequence converging pointwise in $\mathcal{Y}$ to v. Since $(f^{\triangleleft}(v))(x) = v(f(x))$ and $(f^{\triangleleft}(v_n))(x) = v_n(f(x))$, $x \in X, n \in N$, the sequence $\langle f^{\triangleleft}(v_n) \rangle$ converges pointwise in $(X, \mathcal{X})$ to $f^{\triangleleft}(v)$. Thus $f^{\triangleleft}$ is a sequentially continuous map of $(X, \mathcal{Y})$ to $(X, \mathcal{X})$. Second, we have to verify that

a) $f^{\triangleleft}$ sends each distinguished constant of $\mathcal{Y}$ into the corresponding distinguished constant of $\mathcal{X}$: $f^{\triangleleft}(b_{\mathcal{Y}}) = b_{\mathcal{X}}$, $f^{\triangleleft}(l_{\mathcal{Y}}) = l_{\mathcal{X}}$, and $f^{\triangleleft}(r_{\mathcal{Y}}) = r_{\mathcal{X}}$;

b) $f^{\triangleleft}$ preserves the partial order: if $u, v \in \mathcal{Y}$ and $u \leq v$, then $f^{\triangleleft}(u) \leq f^{\triangleleft}(v)$ in $\mathcal{X}$;

c) $f^{\triangleleft}$ preserves the partial operation: if $u, v \in \mathcal{Y}$ and $u \leq v$, then $f^{\triangleleft}(v \oslash u) = f^{\triangleleft}(v) \oslash f^{\triangleleft}(u)$ in $\mathcal{X}$.

Once again, all three conditions follow from the fact that for each $w \in \mathcal{Y}$ and for each $x \in \mathcal{X}$ we have $(f^{\triangleleft}(w))(x) = w(f(x))$. For example, if $u \leq v$ in $\mathcal{Y}$, i.e., $u(y) \leq v(y)$ for all $y \in Y$, then also $(f^{\triangleleft}(u))(x) = u(f(x)) \leq v(f(x)) = (f^{\triangleleft}(v))(x)$ for all $x \in X$, and hence $f^{\triangleleft}(u) \leq f^{\triangleleft}(v)$. Other conditions can be verified analogously.

(ii) Assume that there exist $x \in X$ such that $f(x) \neq g(x)$. Since $\mathcal{Y}$ is reduced, there exists $u \in \mathcal{Y}$ such that $u(f(x)) \neq u(g(x))$. Consequently $(f^{\triangleleft}(u))(x) = u(f(x)) \neq u(g(x)) = (g^{\triangleleft}(u))(x)$ and hence $f^{\triangleleft} \neq g^{\triangleleft}$. □

Definition 3.3. *Let h be a map of an AFD-system $\mathcal{Y}$ into an AFD-system $\mathcal{X}$ preserving the structure of AFD-systems. Then h is said to be an* AFD-homomorphism.

Let $\mathcal{X} \subseteq T^X$ be an AFD-system. Then each $x \in X$ can be considered as a sequentially continuous AFD-homomorphism ev_x of $\mathcal{X}$ into T defined by $ev_x(u) = u(x), u \in \mathcal{X}$. Denote X^* the set of all sequentially continuous AFD-homomorphisms of $\mathcal{X}$ into T. For $u \in \mathcal{X}$ put $u^* = \{ev_x(u);\ x \in X^*\}$ and $\mathcal{X}^* = \{v^*;\ v \in \mathcal{X}\}$. It is easy to see that $\mathcal{X}^*$ is an AFD-system and X^* is the set of all AFD-homomorphisms of $\mathcal{X}^*$ into T. Observe that if $a, b \in \mathcal{X}, a \neq b$, then $ev_a \neq ev_b$. Indeed, $\mathcal{X}$ is reduced and hence $u(a) \neq u(b)$ for some $u \in \mathcal{X}$.

Definition 3.4. *Let $\mathcal{X} \subseteq T^X$ be an AFD-system. Then $\mathcal{X}^*$ is said to be the* sobrification *of $\mathcal{X}$. If $X = X^*$, then $\mathcal{X}$ and $(X, \mathcal{X})$ are said to be* sober.

Theorem 3.5. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces and let h be a sequentially continuous AFD-homomorphism of $\mathcal{Y}$ into $\mathcal{X}$. Then there exists a unique AFD-measurable map f of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ such that $f^{\triangleleft} = h$.*

Proof. For each $x \in X$, the composition $ev_x \circ h$ is a sequentially continuous AFD-homomorphism of $\mathcal{Y}$ into T. Since $\mathcal{Y}$ is reduced and sober, there exists a unique $y \in Y$ such that $ev_y = ev_x \circ h$. Put $y = f(x)$. This defines a map f of X into Y. Let $u \in \mathcal{Y}$. Then for each $x \in X$ we have $(h(u))(x) = = ev_x(h(u)) = (ev_x \circ h)(u) = ev_{f(x)}(u)$. Hence $h = u \circ f = f^{\triangleleft}$. It follows from the preceding lemma that if g is a measurable map of X into Y such that $g^{\triangleleft} = h$, then $f = g$. □

Corollary 3.6. *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces. Then $f \mapsto f^{\triangleleft}$ yields a one-to-one correspondence between $(\mathcal{Y}, \mathcal{X})$-measurable maps and AFD-homomorphisms of $\mathcal{Y}$ to $\mathcal{X}$.*

4. Duality and applications

Denote AFD the category of AFD-systems and sequentially continuous AFD-homomorphisms. Denote SMAFD the subcategory of MAFD consisting of sober AFD-measurable spaces and denote SAFD the corresponding subcategory of AFD consisting of sober objects.

Theorem 4.1. *The categories SMAFD and SAFD are dually isomorphic.*

Proof. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober AFD-measurable spaces. Denote $F((X, \mathcal{X})) = \mathcal{X}$ and $F((Y, \mathcal{Y})) = \mathcal{Y}$. Let f be an AFD-measurable map of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$. Denote $F(f) = f^{\triangleleft}$. A straightforward calculation shows that F yields a contravariant functor of SMAFD into SAFD and that F is a dual isomorphism. □

Observe that the categories AFD and SAFD are isomorphic (indeed, analogously as in [18] it can be proved that $\mathcal{X} \mapsto \mathcal{X}^*$ yields an isomorphism of AFD and SAFD) and, consequently, AFD and SMAFD are dually naturally equivalent. It can be shown that AFD can serve as a base category for antagonistic fuzzy probability theory, AFD-measurable maps can be considered as generalized random variables and their duals can be considered as generalized observables.

References

[1] J. Adámek. Theory of Mathematical Structures. Reidel, Dordrecht, 1983.

[2] K. Atanasov. Intuitionistic Fuzzy Sets: Theory and Applications. Physica Verlag, New York, 1999.

[3] S. Bugajski. Statistical maps I. Basic properties. *Math. Slovaca*, 51, 321–342, 2001.

[4] S. Bugajski. Statistical maps II. Operational random variables. *Math. Slovaca*, 51, 343–361, 2001.

[5] F. Chovanec, F. Kôpka. Difference posets in the quantum structures background. *Int. J. Theoret. Phys.*, 39, 571–583, 2000.

[6] A. Dvurečenskij, S. Pulmannová. New Trends in Quantum Structures. Kluwer Academic Publ., Ister Science, Dordrecht, Bratislava, 2000.

[7] D.J. Foulis, M.K. Bennett. Effect algebras and unsharp quantum logics. *Found. Phys.*, 24, 1331–1352, 1994.

[8] R. Frič. A Stone-type duality and its applications to probability. Topology Proceedings, Vol. 22, Summer 1997, pp. 125-137. Proceedings of the 12th Summer Conference on General Topology and its Applications, North Bay, August 1997.

[9] R. Frič. Convergence and duality. *Appl. Categorical Structures.*, 10, 257–266, 2002.

[10] R. Frič. Duality for generalized events. *Math. Slovaca*, 54, 49–60, 2004.

[11] R. Frič. Remarks on statistical maps and fuzzy (operational) random variables. *Tatra Mountains Math. Publ.*, 30, 21–34, 2005.

[12] R. Frič. Statistical maps: a categorical approach. *Math. Slovaca*, 57, 41–57, 2007.

[13] S. Gudder. Fuzzy probability theory. *Demonstratio Math.*, 31, 235–254, 1998.

[14] H. Herrlich G.E. Strecker. Category Theory. Heldermann Verlag, Berlin, second edition, 1976.

[15] F. Kôpka. D-posets of fuzzy sets. *Tatra Mountains Math. Publ.*, 1, 83–87, 1992.

[16] F. Kôpka, F. Chovanec. D-posets. *Math. Slovaca*, 44, 21–34, 1994.

[17] R. Mesiar. Fuzzy sets and probability theory. *Tatra Mountains Math. Publ.*, 1, 105–123, 1992.

[18] M. Papčo. On measurable spaces and measurable maps. *Tatra Mountains Math. Publ.*, 28, 125–140, 2004.

[19] M. Papčo. On fuzzy random variables: examples and generalizations. *Tatra Mountains Math. Publ.*, 30, 175–185, 2005.

[20] B. Riečan, D. Mundici. Probability on MV-algebras. In: Handbook of Measure Theory, Vol. II (Editor: E. Pap), pp. 869–910, North-Holland, Amsterdam, 2002.

[21] B. Riečan. On a problem of Radko Mesiar: general form of intuitionistic fuzzy probabilities. *Fuzzy Sets and Systems*, 152, 1485–1490, 2006.