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DIDACTIC REMARKS ON THE POWER SET

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Abstract. The paper is devoted to correct understanding of the notation for the power set. Often this notation is mistaken with a power of the number 2. The correct definition of the power set is presented as well as several task which can serve for strengthening the understanding of this notion.

1. Introduction

In the paper [3] the author paid attention to students' erroneous understanding of the notation 2^X. For example, there appear the following understanding of this notation:

$$2^N = \{2^0, 2^1, 2^2, \dots, 2^n, \dots\},$$

$$2^N = \{\{2^0\}, \{2^1\}, \{2^2\}, \dots, \{2^n\}, \dots\},$$

if $X = \{a_1, \dots, a_n\}$, then $2^X = \{\{2^{a_1}\}, \dots, \{2^{a_n}\}\}$.

It seems to us that these mistakes can be easily rectified referring to the following definition:

Definition 1.

$$2^X = \{Y : Y \subseteq X\}.$$

In this case there is no doubt that it is necessary to determine a set (or, more precisely, a family) of all the subsets of the set X.

It is obvious that

$$2^{\emptyset} = \{Y : Y \subseteq \emptyset\} = \{\emptyset\},$$
$$2^{\{a\}} = \{Y : Y \subseteq \{a\}\} = \{\emptyset, \{a\}\},$$
$$2^{\{a,b\}} = \{Y : Y \subseteq \{a, b\}\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

A family $2^{\{a,b\}}$, i.e. a family $\{Y : Y \subseteq 2^{\{a,b\}}\}$, consists of 16 sets.

It should be noted that appearance of the number 2 in the notation 2^X can suggest to somebody that this notation concerns the power of the number 2 or some set of powers of the number 2.

Appearance of the number 2 can be clarified referring to the notion of the set of transformations of one set into another or to the notion of the characteristic function of a given set as well as to the notion of set power or set cardinal number.

A set of all transformations of a set X into a set Y is denoted by the symbol Y^X. The following notations are equivalent:

$$f \in Y^X, \quad f : X \to Y, \quad X \underset{f}{\to} Y.$$

If V denotes a universe, then the characteristic function f_Z of a set Z $(Z \subseteq V))$ is defined as follows:

Definition 2.

$$f_Z(a) = \begin{cases} 1 & \text{if} \quad a \in Z, \\ 0 & \text{if} \quad a \in V - Z. \end{cases}$$

From the power theory (the theory of cardinal numbers) it is known [4] that if $\overline{\overline{X}} = m$, $\overline{\overline{Y}} = n$, where m and n are arbitrary cardinal numbers, then

$$\overline{\overline{Y^X}} = (\overline{\overline{Y}})^{\overline{\overline{X}}} = n^m.$$

If X is an arbitrary set and $Y = \{0, 1\}$, then the power set 2^X is equipotent to the set $\{0, 1\}^X$. Therefore

$$\overline{\overline{2^X}} = 2^{\overline{\overline{X}}} = \left(\left\{\overline{\overline{0, 1}}\right\}\right)^{\overline{\overline{X}}} = \overline{\overline{\{0, 1\}^X}}.$$

If we suppose that $V = X$, then the set $\{0, 1\}^X$ is a set of all the characteristic functions of a set Z contained in X.

Example. If $X = \{a, b\}$, $Y = \{0, 1\}$, then $Y^X = \{0, 1\}^{\{a,b\}} = \{f_1, f_2, f_3, f_4\}$, where

$$\left.\begin{array}{l} f_1(x) = 0 \\ f_2(x) = 1 \end{array}\right\} \text{ for arbitrary } x \in \{a, b\},$$

$$f_3(x) = \begin{cases} 1 & \text{if} \quad x = a, \\ 0 & \text{if} \quad x = b, \end{cases}$$

$$f_4(x) = \begin{cases} 0 & \text{if} \quad x = a, \\ 1 & \text{if} \quad x = b. \end{cases}$$

Then $\overline{\overline{Y^X}} = \overline{\overline{\{0,1\}^{\{a,b\}}}} = \left(\{\overline{\overline{0,1}}\}\right)^{\overline{\overline{\{a,b\}}}} = 2^2 = 4^{\overline{\overline{X}}}$.

The functions f_1, f_2, f_3, f_4 are the characteristic functions of the sets $\emptyset$, $\{a,b\}$, $\{a\}$, $\{b\}$, respectively.

Suppose that a set X is finite and contains n elements ($\overline{\overline{X}} = n$). Calculating the number of all the subsets of a set X, we obtain:

$\binom{n}{0}$ – the number of empty sets $\left(\binom{n}{0} = 1\right)$,

$\binom{n}{1}$ – the number of one-element sets,

$\binom{n}{2}$ – the number of two-element sets,

. . . .

$\binom{n}{n}$ – the number of n-element sets $\left(\binom{n}{n} = 1\right)$.

From the Newton formula $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for $x = 1$, $y = 1$ we get $\sum\limits_{k=0}^{n} \binom{n}{k} = 2^n$. Therefore, the number $2^{\overline{\overline{X}}}$ is equal to the number of all the subsets of a set X.

It can be assumed that some troubles related to set theory appear if we speak about family of sets or about family of family of sets. This is connected with the fact that in the consciousness of many people there are two conceptions of a set [1, 2]:

– in distributive sense (set-theoretic),

– in collective sense (mereological).

For example, in distributive understanding of the notion of a set a one-elemnt set is identified with its element: $(\{a\} = a,\ \{\{a\}\} = \{a\})$.

Sometimes, the features of operations on numbers are extended to set-theory operations. For example, there appear the following equalities:

$$2^A \cap 2^B = 2^{A \cup B}, \quad (A \cup B) - B = A, \quad B \cup (A - B) = A,$$

but the first of these equalities is valid when $A = B$, the second one when $A \cap B = \emptyset$, and the third when $B \subseteq A$.

The task on finding the following sets can serve for strengthening the understanding of the notion of power set:

1. $\inf(2^X, 2^Y)$, $\sup(2^X, 2^Y)$,
2. $2^X \cap 2^Y$, $2^X \cup 2^Y$, $2^X - 2^Y$, $2^X \dot{-} 2^Y$,
   where $X = \{a, b, c\}$, $Y = \{b, c, d\}$, and " $\dot{-}$ " is the operation of symmetric difference: $(A \dot{-} B = (A - B) \cup (B - A))$,
3. $2^{\{a\}} - \{a\}$, $\quad 2^{2^{\{a\}}} - 2^{\{a\}}$,
4. $2^X \cap 2^{V-X}$, where $V = \{a_1, a_2, \ldots, a_n\}$, $X = \{a_1, a_2, \ldots, a_{n-1}\}$.

References

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