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REMARKS ON SOLUTIONS OF EQUATIONS WITH ABSOLUTE VALUE FUNCTION

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Abstract. In this paper, we give the proposition of didactic discussion on the existence of solutions of some equations containing expressions with absolute value. We are interested in the possibility of applying the method of conceptual problem-solving. Under this term we understand such a procedure, in which the solver is not limited to the automatic application of the definition of the absolute value, but he can reject some cases based on his mathematical knowledge. To do this, one must make use from various features of this concept – not only from its definition.

1. Introduction

In the paper [1], we included remarks on solving several types of equations containing terms with the absolute value. There we paid attention on the conceptual problem-solving, i.e. the procedure of solving based on the use of theorems which involve the absolute value. We also set this method against the algorithmic method of solving, i.e. the procedure which bases on mechanical use of absolute value definition and considering cases deriving from the range of formulas' applicability.

In the current paper, we present some examples of problems which are generated by the discussion of existence of solutions to the following equation:

$$|f(x)| + |g(x)| = m, \qquad (1)$$

where $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ and m is an arbitrary real number. We are interested in the possibility of applying the conceptual problem-solving method in reference to equations of this type.

Issues listed below may provide a basis for building mathematical tasks and problems which, thanks to the use of conceptual problem-solving method, allow us to intense creative mathematical activity.

2. Exemplary issues

First, let us notice that equation (1) has a solution only if m belongs to the set of values of the function $x \to |f(x)| + |g(x)|$. However, indication of this set sometimes is importunate. The form of equation (1) ensues the following prerequisite for existence of solution to this equation:

Theorem 1. *Let $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ with $D_1 \cap D_2 \neq \emptyset$ be given functions and let m be an arbitrary real number. If equation (1) has a solution, then m is a nonnegative number.*

Non-negativity of the number m is not the sufficient condition for existence of the solution to equation (1). This is evidenced by the following example.

Example 1. *Let us consider the equation*

$$|x^2 - 4| + |3x| = m.$$

The analysis of the graph of the function $x \to |x^2 - 4| + |3x|$ for $x \in R$ demonstrates that the values of the function are obviously nonnegative. We can also notice that the discussed equation does not have solutions for $0 \leq m < 4$.

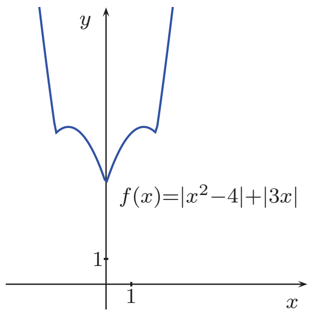

Figure 1.

To find the sufficient conditions for existence of a solution we start with the proof of the following lemma.

Lemma 1. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$. Let A and B designate the following sets:*

$$A = \{x \in D_1 \cap D_2 : f(x)g(x) \geq 0\}, \quad B = \{x \in D_1 \cap D_2 : f(x)g(x) < 0\}.$$

Therefore,

$$\max(|f(x) + g(x)|, |f(x) - g(x)|) = \begin{cases} |f(x) + g(x)|, & x \in A, \\ |f(x) - g(x)|, & x \in B. \end{cases}$$

Proof. Let $x \in A$. Then $|f(x)+g(x)| = f(x)+g(x)$, when $f(x)$ and $g(x)$ are nonnegative, or $|f(x)+g(x)| = -f(x)-g(x)$, when $f(x)$ and $g(x)$ are negative, and $|f(x)-g(x)| = f(x)-g(x)$, when $f(x) \geq g(x)$ or $|f(x)-g(x)| = -f(x)+g(x)$, when $f(x) < g(x)$. Hence, it follows that $\max(|f(x)+g(x)|, |f(x)-g(x)|) = |f(x)+g(x)|$.

For $x \in B$ we have

$$|f(x)+g(x)| = \begin{cases} f(x)+g(x), & x \in B \text{ and } f(x)+g(x) \geq 0, \\ -f(x)-g(x), & x \in B \text{ and } f(x)+g(x) < 0, \end{cases}$$

and

$$|f(x)-g(x)| = \begin{cases} f(x)-g(x), & x \in B \text{ and } f(x)-g(x) \geq 0, \\ -f(x)+g(x), & x \in B \text{ and } f(x)-g(x) < 0. \end{cases}$$

Therefore, it ensues that $\max(|f(x)+g(x)|, |f(x)-g(x)|| = |f(x)-g(x)|$, which ends the proof.

In posterior deliberations we will make use of the known feature of the absolute value of real number.

Lemma 2. *For arbitrary real numbers p, q the following equation is satisfied*

$$|p| + |q| = \max(|p+q|, |p-q|).$$

The following theorem is true.

Theorem 2. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ be given functions and let m be an arbitrary nonnegative real number. Equation* (1) *has a solution if and only if there exists the number $x_0 \in D_1 \cap D_2$ such that the following conditions hold*

$$|f(x_0)+g(x_0)| = m \quad \textit{and} \quad |f(x_0)-g(x_0)| = m,$$

or

$$|f(x_0)+g(x_0)| = m \quad \textit{and} \quad |f(x_0)-g(x_0)| < m,$$

or

$$|f(x_0)+g(x_0)| < m \quad \textit{and} \quad |f(x_0)-g(x_0)| = m.$$

Proof. If equation (1) has a solution, then there exists the number $x_0 \in D_1 \cap D_2$ such that

$$|f(x_0)| + |g(x_0)| = m.$$

It follows herefrom and from lemma 2 that

$$\max(|f(x_0)+g(x_0)|, |f(x_0)-g(x_0)|) = m, \tag{2}$$

and, in consequence, conditions contained in the thesis of the led theorem.

Let us presume that there exists $x_0 \in D_1 \cap D_2$ which satisfies disjunction of conditions from the led theorem. Thus, equation (2) holds and on account of lemma 2 we receive that x_0 is the solution of equation (1).

In the quoted paper [1], we consider equation (1) in which functions f and g satisfy the condition

$$\exists_{c\in\mathbb{R}}\forall_{x\in D_1\cap D_2}|f(x)-g(x)|=c,$$

and the function $|f(x)+g(x)|$ is boundless from the top in its domain. The below theorem is the generalization of theorem 5 from the mentioned paper.

Theorem 3. *Let* $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ *and* $D_1 \cap D_2 \neq \emptyset$ *be given functions,* $h_1(x) = |f(x)+g(x)|$, $h_2(x) = |f(x)-g(x)|$, $x \in D_1 \cap D_2$ *and let* m *be an arbitrary nonnegative real number.*

a) If

$$\exists_{c\in\mathbb{R}^+}\forall_{x\in D_1\cap D_2}|f(x)-g(x)|=c \tag{3}$$

and $m \in h_1(D_1 \cap D_2)$, *hence equation* (1) *has a solution if and only if*

$$c \leq m. \tag{4}$$

b) If

$$\exists_{c\in\mathbb{R}^+}\forall_{x\in D_1\cap D_2}|f(x)+g(x)|=c \tag{5}$$

and $m \in h_2(D_1 \cap D_2)$, *hence equation* (1) *has a solution if and only if the inequality (4) holds.*

Proof.

a) Let us assume that equation (1) has the solution $x_0 \in D_1 \cap D_2$. Then, by lemma 2 and condition (3), we have

$$\max(|f(x_0)+g(x_0)|, c) = m,$$

hence inequality (4) holds. Let us assume that inequality (4) holds. From (1), lemmas 2 and (3) we have

$$|f(x)|+|g(x)| = \max(|f(x)+g(x)|, c) = m.$$

Hence, from (4) we receive the inequality $|f(x)+g(x)| \leq m$. The fact that $m \in h_1(D_1 \cap D_2)$ shows that there exists the number $x_0 \in D_1 \cap D_2$ which satisfies equation (1).

b) The proof of this part of the theorem is carried out in analogical fashion.

Assuming (3) or (5), it follows from theorem 3 that equation (1) can be replaced by the equation

$$|f(x) - g(x)| = m \quad \text{or} \quad |f(x) + g(x)| = m.$$

Let us consider the following example.

Example 2.

a) The equation $|ax+b|+|ax+c| = m$, where a, b, c, and m are given real numbers, has a solution on the strength of theorem 3a) if and only if $|b - c| \le m$. Thus, it can be replaced by the equation $|2ax + b + c| = m$ (compare [1]).

b) The equation $|-x^2| + |-x^2 + 2| = m$ has a solution on the strength of theorem 3b) if and only if $m \ge 2$. Thus, it can be replaced by the equation $|-2x^2 - 2| = m$.

c) Theorem 3 does not determinate the number of solutions of equation 1. For example, the equation $|\sin x| + |\sin x + 2| = m$ has infinitely many solutions for $m = 2$ (see Figure 2).

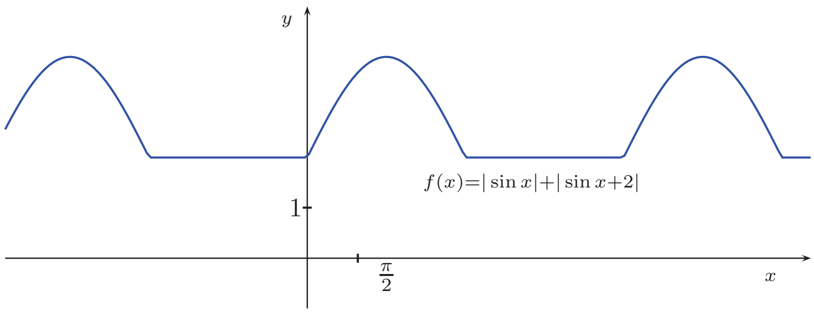

Figure 2.

In Figure 2, the graph of the function $f(x) = |\sin x| + |\sin x + 2|$ is presented. The condition for existence of solution to the considered equation cannot be described by means of theorem 3, because the absolute values of the sum and the difference of the functions are not boundless. Simultaneously, this equation can be replaced by the equation $|2\sin(x) + 2| = m$ if $m \in [2, 4]$.

At the end we will solve the problem of existence of solutions to equation (1) with additional assumption.

Theorem 4. *Let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and $D_1 \cap D_2 \neq \emptyset$ be given functions and let m and k be arbitrary positive real numbers. Equation* (1) *has a solution which satisfies the condition*

$$f^2(x) + g^2(x) = k^2 \tag{6}$$

if and only if $m \in [k, k\sqrt{2}]$.

Proof. From (6) and (1) we obtain the equation

$$|f(x)| + \sqrt{k^2 - f^2(x)} = m,$$

assuming that $|f(x)| \leq k$. Hence we get the equation

$$2f^2(x) - 2m|f(x)| + m^2 - k^2 = 0,$$

which has a solution for $m \leq k\sqrt{2}$. Moreover, if we square equation (1) and take into account condition (6), then we obtain that $m \geq k$, which ends the proof of the theorem.

Let us notice that if $f(x) = \sin x$ and $g(x) = \cos x$, then $k = 1$. The following conclusion is derived from theorem 4.

Conclusion 1. *The equation*

$$|\sin x| + |\cos x| = m$$

has a solutions if and only if $m \in [1, \sqrt{2}]$.

From conslusion 1 it follows that the set $[1, \sqrt{2}]$ is the set of values of the function $x \to |\sin x| + |\cos x|$. This fact may explain the frequent appearance of the following task in many of the tasks collections:

Solve the equation

$$|\sin^2 x| + |\cos^2 x| = \sqrt{2}.$$

3. Summary

Issues presented in this paper and some similar issues were considered at class with students of Mathematics Teaching Faculty. Observations of students' work and researches carried out in other groups (see [1]) indicate that there occur great difficulties of learners in formulation of hypotheses, including the necessary conditions and sufficient conditions for relevant facts. Surveyed students have considered equations mainly by putting particular values into formulas. This type of attitude can be explained by some mathematical immaturity of students in the field of general mathematical reasonings.

References

[1] J. Major, Z. Powązka. Pewne problemy dydaktyczne związane z pojęciem wartości bezwzględnej. *Annales Academiae Paedagogicae Cracoviensis Studia Ad Didacticam Mathematicae Pertinentia* I, 36, 163-185, 2006.