Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

A CHARACTERIZATION OF A HOMOGRAPHIC TYPE FUNCTION II

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Abstract. This article is a continuation of the investigations contained in the previous paper [2]. We deal with the following conditional functional equation:

$$f(x)f(y) \neq \frac{1}{\lambda^2} \quad \text{implies} \quad f(x \star y) = \frac{f(x)+f(y)+2\lambda f(x)f(y)}{1-\lambda^2 f(x)f(y)}$$

with $\lambda \neq 0$.

1. Introduction

If $(G, \star)$ is a group or a semigroup and F stands for an arbitrary binary operation in some set H, then a solution of the functional equation

$$f(x \star y) = F(f(x), f(y))$$

is called a homomorphism of structures $(G, \star)$ and (H, F).

Let $J \subset \mathbb{R}$ be a nontrivial interval and $I \subset \mathbb{R}$ be an interval such that $I + I \subset I$. Let further $F : J \times J \longrightarrow J$ be a given map. Functional equations of the form

$$f(x+y) = F(f(x), f(y)), \quad x, y \in I,$$

have nonconstant continuous solutions if and only if there exists an open interval constituting a continuous group with respect to the associative operation F. All such solutions are strictly monotonic (see Aczél [1]). Here we consider a rational function $F : \{(x,y) \in \mathbb{R} : \quad xy \neq \frac{1}{\lambda^2}\} \longrightarrow \mathbb{R}$ of the form

$$F(u,v) = \frac{u+v+2\lambda uv}{1-\lambda^2 uv}$$

with $\lambda \neq 0$. This is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$ that with every $\lambda \in \mathbb{R} \setminus \{0\}$ satisfies the equation

$$F(F(x,y),z) = F(x,F(y,z))$$

for all $(x,y,z) \in \mathbb{R}^3$ such that products $xy, yz, F(x,y)z, xF(y,z)$ are not equal to λ^{-2}. Rational functions with such or similar properties are termed associative operations.

A homografic function $\varphi : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(x) = \frac{x}{\lambda - \lambda x}, \quad x \neq 1,$$

satisfies the functional equation

$$f(x+y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}$$

for every pair $(x,y) \in \mathbb{R}^2 \setminus D$, where

$$D = \{(x, 1-x) : \quad x \in \mathbb{R}\} \cup \{(x,1) : \quad x \in \mathbb{R}\} \cup \{(1,x) : \quad x \in \mathbb{R}\}.$$

We shall determine all the functions $f : G \longrightarrow \mathbb{R}$, where $(G, \star)$ is a group, that satisfy the functional equation

$$f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}. \tag{1}$$

By a solution of the functional equation (1) we understand any function $f : G \longrightarrow \mathbb{R}$ that satisfies the equality (1) for every pair $(x,y) \in G^2$ such that $f(x)f(y) \neq \lambda^{-2}$. Thus we deal with the following conditional functional equation:

$$f(x)f(y) \neq \frac{1}{\lambda^2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)} \tag{E}$$

for all $x, y \in G$.

The solution of equation (E) in the case $\lambda = 1$ was described in [2].

2. Main result

We proceed with a description of solutions of (E).

Theorem. *Let $(G, \star)$ be a group and $\lambda \in \mathbb{R} \setminus \{0\}$ be fixed. A function $f : G \longrightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation*

$$f(x)f(y) \neq \lambda^{-2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)} \tag{E}$$

for all $x, y \in G$ if and only if either

$$f(x) := \begin{cases} \frac{1}{\lambda} & \textit{for } x \in H, \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus H \end{cases}$$

or

$$f(x) := \begin{cases} \frac{A(x)}{\lambda - \lambda A(x)} & \textit{for } x \in \Gamma \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus \Gamma \end{cases}$$

or

$$f(x) := \begin{cases} \frac{1}{\lambda} & \textit{for } x \in \Gamma \setminus Z \\ 0 & \textit{for } x \in Z \\ -\frac{1}{\lambda} & \textit{for } x \in G \setminus \Gamma, \end{cases}$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, $(Z, \star)$ is a subgroup of the group $(\Gamma, \star)$, and $A : \Gamma \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(\Gamma)$.

Proof. Assume that f is a nonconstant solution of equation (E), i.e.

$$f(x)f(y) \neq \lambda^{-2} \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2\lambda f(x)f(y)}{1 - \lambda^2 f(x)f(y)}$$

for all $x, y \in G$. Hence

$$\lambda^2 f(x)f(y) \neq 1 \quad \text{implies} \quad \lambda f(x \star y) = \frac{\lambda f(x) + \lambda f(y) + 2\lambda^2 f(x)f(y)}{1 - \lambda^2 f(x)f(y)}.$$

Thus, it is easy to obserwe that (E) states that the function $g := \lambda f$ satisfies the following functional equation:

$$g(x)g(y) \neq 1 \quad \text{implies} \quad g(x \star y) = \frac{g(x) + g(y) + 2g(x)g(y)}{1 - g(x)g(y)}$$

for all $x, y \in G$. From the theorem proved by the author in [2] we conclude that g is of the form

$$g(x) := \begin{cases} 1 & \text{for } x \in H, \\ -1 & \text{for } x \in G \setminus H \end{cases}$$

or

$$g(x) := \begin{cases} \frac{A(x)}{1 - A(x)} & \text{for } x \in \Gamma \\ -1 & \text{for } x \in G \setminus \Gamma \end{cases}$$

or

$$g(x) := \begin{cases} 1 & \text{for } x \in \Gamma \setminus Z \\ 0 & \text{for } x \in Z \\ -1 & \text{for } x \in G \setminus \Gamma, \end{cases}$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, $(Z, \star)$ is a subgroup of the group $(\Gamma, \star)$, and $A : \Gamma \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(\Gamma)$. This means that f is of the form as above.

It is easy to check that each of the functions above yields a solution to the equation (E). Thus the proof has been completed.

The following remark gives the form of constant solutions to equation (E).

Remark. *Let $(G, \star)$ be a group. The only constant solutions of equation* (E) *are $f = -\frac{1}{\lambda}, f = 0$ and $f = \frac{1}{\lambda}$.*

To check this, assume that $f = c$ fulfils (E). Then

$$c^2 \neq \frac{1}{\lambda^2} \Longrightarrow c = 2c\frac{1 + \lambda c}{1 - \lambda^2 c^2},$$

i.e.

$$c \in \left\{-\frac{1}{\lambda}, \frac{1}{\lambda}\right\} \quad \text{or} \quad c = 0 \quad \text{or} \quad 1 = 2\frac{1 + \lambda c}{1 - \lambda^2 c^2},$$

whence

$$c \in \left\{-\frac{1}{\lambda}, 0, \frac{1}{\lambda}\right\},$$

which was to be shown.

Remark. *Solutions of* (1) *for $\lambda \in \{-1, 1\}$ in the class of continuous functions can be found in* [1].

References

[1] J. Aczél. *Lectures on Functional Equations and Their Applications.* Academic Press, New York 1966.

[2] K. Domańska. A characterization of a homographic type function. *Scientific Issues, Jan Długosz University in Częstochowa, Mathematics*, **XV**, 25–30, 2010.