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THE STABILITY OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract

In the present paper we deal with the Dhombres-type trigonometric difference

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\left[f(y) + g(y)\right],$$

assuming that its absolute value is majorized by some constant. Our aim is to find functions $\widetilde{f}$ and $\widetilde{g}$ which satisfy the Dhombres-type trigonometric functional equation and for which the differences $\widetilde{f} - f$ and $\widetilde{g} - g$ are uniformly bounded.

1. Introduction

Stability problems concerning classical functional equations have been considered by several authors (see, e.g., [5]–[7]). The cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1}$$

and the sine functional equation

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \tag{2}$$

are both stable (in fact, they are even superstable) in the Hyers-Ulam sense. In [3], Baker studied the stability of the cosine functional equation (1), while Cholewa established the stability of the sine functional equation (2) in [4]. The results about the superstability can be obtained as corollaries from theorems by Badora and Ger. Namely, the following theorems hold.

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Theorem 1 (Badora and Ger, see [2]). *Let $(G,+)$ be an Abelian group and let $f\colon G\to\mathbb{C}$ and $\varphi\colon G\to\mathbb{R}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)f(y)\big|\le\varphi(x)\quad\text{for all}\quad x,y\in G.$$

Then either f is bounded or

$$f(x+y)+f(x-y)=2f(x)f(y)\quad\text{for all}\quad x,y\in G.$$

Theorem 2 (Badora and Ger, see [2]). *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $f\colon G\to\mathbb{C}$ and $\varphi\colon G\to\mathbb{R}$ satisfy the inequality*

$$\left|f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\le\varphi(x)\quad\text{for all}\quad x,y\in G.$$

Then either f is bounded or

$$f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2\quad\text{for all}\quad x,y\in G.$$

From now on, we denote the odd and the even parts of a function f by f_o and f_e, respectively. The next lemma (due to Wilson, see [11]) provides general solutions of an equation that generalizes the equation (1).

Lemma 1 (see also [1] and [8]). *Let $(G,+)$ be an Abelian group. Then functions $f,g\colon G\to\mathbb{C}$ satisfy the functional equation*

$$f(x+y)+f(x-y)=2f(x)g(y)\tag{3}$$

if and only if one of the following conditions holds:

(i) *the function g is arbitrary and $f=0$;*

(ii) *there exist an additive function $a\colon G\to\mathbb{C}$ and a constant $\alpha\in\mathbb{C}$ such that*

$$f(x)=a(x)+\alpha\quad\text{and}\quad g(x)=1\quad\text{for all}\quad x\in G;$$

(iii) *there exist an exponential function $m\colon G\to\mathbb{C}$ and constants $\beta,\gamma\in\mathbb{C}$ such that*

$$f(x)=\beta m_o(x)+\gamma m_e(x)\quad\text{and}\quad g(x)=m_e(x)\quad\text{for all}\quad x\in G.$$

In [8], Székelyhidi studied the Hyers-Ulam stability of the equation (3), obtaining the following result.

Theorem 3. *Let $(G,+)$ be an Abelian group and let $\varepsilon \geq 0$. If functions $f,g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)g(y)\big| \leq \varepsilon \quad \text{for all} \quad x,y \in G,$$

then one of the following conditions holds:

(i) *if $f=0$, then g is arbitrary;*

(ii) *if $f \neq 0$ is bounded, then g is bounded, as well;*

(iii) *if g is bounded and f is unbounded, then $g=1$ and there exist an additive function $A\colon G \to \mathbb{C}$ and a constant $\delta \in \mathbb{C}$ such that*

$$\big|f(x)-A(x)\big| \leq \delta \quad \text{for all} \quad x \in G;$$

(iv) *if $f \neq 0$ and g is unbounded, then f is unbounded, as well. Moreover, functions f and g satisfy the equation* (3).

The above theorems allow us to formulate the following result concerning superstability.

Corollary. *Let unbounded functions $f,g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)g(y)\big| \leq \varepsilon$$

for all $x,y \in G$ and for some $\varepsilon \geq 0$. Then f and g satisfy the equation (3).

The aim of this paper is to study stability properties of the Dhombres-type trigonometric functional equation, i.e.,

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y)+f(x-y) = f(x)\big[f(y)+g(y)\big]. \tag{4}$$

In the case when g is slightly specialized solutions of the above equation can be found in [10].

We shall use the following lemma.

Lemma 2 (see [9, Corollary 3])**.** *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $\varepsilon \geq 0$. Let an unbounded function $f\colon G \to \mathbb{C}$ and a function $g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)g(y)\right| \leq \varepsilon \quad \text{for all} \quad x,y \in G.$$

Then one of the following conditions holds:

(i) *if $g \neq 0$ is bounded, then g satisfies the sine equation* (2)*;*

(ii) *if g is unbounded, then there exists a function $h\colon G \to \mathbb{C}$ such that*

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x, y \in G.$$

For $f = f_e + f_o$, we have $f_e(x) = f(0)h(x)$ for all $x \in G$ and f_o satisfies the sine equation. Moreover, if $f(0) = 0$, then $g = f = f_o$.

2. Main result

Our main result reads as follows.

Theorem 4. *Let $(G, +)$ be a uniquely 2-divisible Abelian group and let $\varepsilon \geq 0$. If functions $f, g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[f(y) + g(y)\big] \right| \leq \varepsilon \tag{5}$$

for all $x, y \in G$, then there exist an exponential function $m\colon G \to \mathbb{C}$, additive functions $a, A\colon G \to \mathbb{C}$, a bounded function $B\colon G \to \mathbb{C}$ and a constant β such that one of the following conditions holds:

(i) *if $f = 0$, then g is arbitrary;*
(ii) *if $f \neq 0$ is bounded, then g is bounded, as well;*
(iii) *if the function f is unbounded, then*

$$\begin{cases} f(x) = A(x) + B(x) \\ g(x) = a(x) - A(x) - B(x) + 2 \end{cases} \quad \text{for all} \quad x \in G, \tag{6}$$

or

$$\begin{cases} f(x) = f_o(x) + f(0)m_e(x) \\ g(x) = \big(\beta m_o(x) - f_o(x)\big) + \big(2 - f(0)\big)m_e(x) \end{cases} \quad \text{for all} \quad x \in G. \tag{7}$$

Moreover, suppose that $f(0) = 0$. Then

$$\begin{cases} f(x) = \beta m_o(x) \\ g(x) = 2m_e(x) \end{cases} \quad \text{for all} \quad x \in G. \tag{8}$$

Proof. Assume that the function f is an unbounded solution of inequality (5). Then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ of elements of G such that

$$0 \neq |f(z_n)| \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{9}$$

Let us take $x = z_n$ in (5). Then we obtain

$$\left| f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y) - f(z_n)\big[f(y)+g(y)\big] \right| \le \varepsilon$$

for all $y \in G$ and $n \in \mathbb{N}$, whence

$$\left| \frac{f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y)}{f(z_n)} - \big[f(y)+g(y)\big] \right| \le \frac{\varepsilon}{|f(z_n)|}$$

for all $y \in G$ and $n \in \mathbb{N}$. Now, taking the limit as $n \to \infty$ and applying (9), we obtain

$$\lim_{n\to\infty} \frac{f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y)}{f(z_n)} = f(y) + g(y) \tag{10}$$

for all $y \in G$. Hence,

$$f(0) + g(0) = 2. \tag{11}$$

Let us replace x by $z_n + x$ in (5). Then we get

$$\left| f\left(\frac{z_n+x+y}{2}\right)^2 - f\left(\frac{z_n+x-y}{2}\right)^2 + f(z_n+x+y) \right.$$
$$\left. + f(z_n+x-y) - f(z_n+x)\big[f(y)+g(y)\big] \right| \le \varepsilon.$$

Similarly, let us replace x by $z_n - x$ in (5). Then

$$\left| f\left(\frac{z_n-x+y}{2}\right)^2 - f\left(\frac{z_n-x-y}{2}\right)^2 + f(z_n-x+y) \right.$$
$$\left. + f(z_n-x-y) - f(z_n-x)[f(y)+g(y)] \right| \le \varepsilon.$$

From the above inequalities we compute

$$\left| f\left(\frac{z_n+(x+y)}{2}\right)^2 - f\left(\frac{z_n-(x+y)}{2}\right)^2 + f\big(z_n+(x+y)\big) \right.$$
$$+ f\big(z_n-(x+y)\big) + f\left(\frac{z_n+(-x+y)}{2}\right)^2 - f\left(\frac{z_n-(-x+y)}{2}\right)^2$$
$$+ f\big(z_n+(-x+y)\big) + f\big(z_n-(-x+y)\big)$$
$$\left. - \big[f(z_n+x)+f(z_n-x)\big]\cdot\big[f(y)+g(y)\big] \right| \le \varepsilon$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Therefore,

$$\left| \frac{f\left(\frac{z_n+(x+y)}{2}\right)^2 - f\left(\frac{z_n-(x+y)}{2}\right)^2 + f(z_n+(x+y)) + f(z_n-(x+y))}{f(z_n)} \right.$$
$$+ \frac{f\left(\frac{z_n+(-x+y)}{2}\right)^2 - f\left(\frac{z_n-(-x+y)}{2}\right)^2 + f(z_n+(-x+y)) + f(z_n-(-x+y))}{f(z_n)}$$
$$\left. - \frac{[f(z_n+x)+f(z_n-x)]}{f(z_n)}\big[f(y)+g(y)\big] \right| \le \varepsilon$$

for all $x, y \in G$ and $n \in \mathbb{N}$. With the use of (9) and (10), we conclude that for every $x \in G$ there exists the following limit as n tends to infinity:

$$\lim_{n\to\infty} \frac{f(z_n+x)+f(z_n-x)}{f(z_n)} =: h(x). \tag{12}$$

Moreover, the so defined function $h\colon G \to \mathbb{C}$ satisfies the equation

$$f(x+y)+g(x+y)+f(-x+y)+g(-x+y)-h(x)\big[f(y)+g(y)\big] = 0, \quad x, y \in G.$$

By interchanging x and y, we obtain

$$f(x+y)+g(x+y)+f(x-y)+g(x-y)-\big[f(x)+g(x)\big]h(y) = 0, \quad x, y \in G.$$

Let $F := f + g$ and $G := \frac{1}{2}h$. Then

$$F(x+y)+F(x-y) = 2F(x)G(y) \quad \text{for all} \quad x, y \in G.$$

On the basis of Lemma 1, we get three possible forms of the function F.

Case 1. Suppose $F = 0$. By putting $f + g = 0$ in (5), we obtain

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) \right| \le \varepsilon$$

for all $x, y \in G$. Now, setting $y = 0$, we get

$$\big|f(x)\big| \le \frac{\varepsilon}{2} \quad \text{for all} \quad x \in G.$$

This leads to a contradiction since f is unbounded.

Case 2. By Lemma 1 case (ii), let us assume that there exist an additive function a and a constant α such that $F = a + \alpha$. Then

$$f(x)+g(x) = a(x)+\alpha \quad \text{for all} \quad x \in G. \tag{13}$$

Let us take $x = 0$ in (13) and apply (11). We get $\alpha = 2$. By the inequality (5), we obtain

$$(14)\quad \left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[a(y)+2\big] \right| \le \varepsilon$$

for all $x, y \in G$. By taking $-y$ instead of y in (14), we infer that

$$(15)\quad \left| f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-a(y)+2\big] \right| \le \varepsilon.$$

By (14) and (15) and the fact that a is odd, we get the following relation:

$$\Bigg| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[a(y)+2\big]$$
$$+ f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-a(y)+2\big] \Bigg| \le 2\varepsilon$$

for all $x, y \in G$. Equivalently,

$$\big|f(x+y) + f(x-y) - 2f(x)\big| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

Applying Theorem 3 to the unbounded function f yields that there exist an additive function A and a constant δ such that

$$\big|f(x) - A(x)\big| \le \delta \quad \text{for all} \quad x, y \in G,$$

hence $f = A + B$, where the function B is bounded by δ. By this fact and by (13) we get $g = a - A - B + 2$. Finally, we obtain (6).

Case 3. By case (iii) of Lemma 1, let us consider $F = \beta m_o + \gamma m_e$, where the function m is exponential and β, γ are constants. Hence, from (5), we obtain

$$(16)\qquad f(x) + g(x) = \beta m_o(x) + \gamma m_e(x) \quad \text{for all} \quad x \in G.$$

Applying (16) to $x = 0$ in (11), we get $\gamma m_e(0) = 2$. We know that if $m \neq 0$, then $m(0) = 1$ and $m_e(0) = 1$. In the other words, we have $\gamma = 2$. By (16) and (5), we get the following relation:

$$(17)\quad \left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[\beta m_o(y) + 2m_e(y)\big] \right| \le \varepsilon$$

for all $x, y \in G$. Hence, by replacing y by $-y$ in (17), we see that

$$(18)\quad \left| f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-\beta m_o(y) + 2m_e(y)\big] \right| \le \varepsilon$$

for all $x, y \in G$. Summing inequalities (17) and (18) sidewise, we infer that

$$\left|f(x+y)+f(x-y)-2f(x)m_e(y)\right| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

If $m_e = 1$, then we obtain Case 2. Therefore, by Theorem 3, functions f and m_e satisfy the following equation:

$$f(x+y)+f(x-y)=2f(x)m_e(y) \quad \text{for all} \quad x \in G. \tag{19}$$

Applying (19) to (17), we see that

$$\left| f\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 + 2f(x)m_e(y) - f(x)\big[\beta m_o(y) - 2m_e(y)\big] \right| \le \varepsilon,$$

for all $x, y \in G$, which is equivalent to

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x)\beta m_o(y) \right| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

From Lemma 2 we get what follows.

Subcase 3.1 If the function βm_o is bounded, then from (i) we only conclude that our function satisfies the sine functional equation.

Subcase 3.2 Assume that the function βm_o is unbounded. Therefore, by (ii), there exists a function h such that

$$f(x+y)+f(x-y)=2f(x)h(y) \quad \text{for all} \quad x \in G. \tag{20}$$

Thus, from equations (19) and (20), we obtain $h = m_e$. Furthermore, we get $f = f(0)m_e + f_o$ and the function f_o satisfies the sine equation (2). It follows from (16) that

$$f(0)m_e(x) + f_o(x) + g(x) = \beta m_o(x) + 2m_e(x) \quad \text{for all} \quad x \in G.$$

Equivalently,

$$g(x) = \beta m_o(x) - f_o(x) + \big(2 - f(0)\big)m_e(x) \quad \text{for all} \quad x \in G. \tag{21}$$

Thus, we have proved that functions f and g are of the form (7).

Moreover, by Lemma 2 applied to a function f for which $f(0) = 0$, we have $\beta m_o = f = f_o$. Hence, the above considerations and the equation (21) imply that $g = 2m_e$. The proof of the theorem is complete. $\square$

Assume that an unbounded function $f\colon G \to \mathbb{C}$ such that $f(0) = 0$ and a function $g\colon G \to \mathbb{C}$ satisfy the inequality (5) for all $x, y \in G$. We ask whether there exist functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ and a constant δ such that $\widetilde{f}$ and $\widetilde{g}$ satisfy the equation (4) and

$$\bigl|\widetilde{f}(x) - f(x)\bigr| \le \delta \quad \text{and} \quad \bigl|\widetilde{g}(x) - g(x)\bigr| \le \delta \quad \text{for all} \quad x \in G.$$

By Theorem 4, functions f and g have one of the forms (6) or (8).

Case 1. In the case of the form (8), provided $f(0) = 0$, we define functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ by $\widetilde{f} := f$ and $\widetilde{g} := g$. Therefore, for all $x, y \in G$, we get

$$\beta^2 \left[m_o \left(\frac{x+y}{2} \right)^2 - m_o \left(\frac{x-y}{2} \right)^2 \right] + \beta \left[m_o(x+y) + m_o(x-y) \right] = \beta m_o(x) \bigl[\beta m_o(y) + 2m_e(y) \bigr].$$

Case 2. In the case of the form (6), we have two possibilities.

Subcase 2.1. The function g is bounded. Then the function $a - A$ is bounded and additive. Therefore, $a - A = 0$. Let us define functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ by

$$\begin{cases} \widetilde{f}(x) := A(x) \\ \widetilde{g}(x) := 2 \end{cases} \quad \text{for all} \quad x \in G.$$

Then the so defined functions satisfy the equation (4), i.e.,

$$A \left(\frac{x+y}{2} \right)^2 - A \left(\frac{x-y}{2} \right)^2 + A(x+y) + A(x-y) = A(x) \bigl[A(y) + 2 \bigr]$$

for all $x, y \in G$, and $\bigl|\widetilde{f} - f\bigr| = |B| \le \delta$ and $\bigl|\widetilde{g} - g\bigr| = |B| \le \delta$ for some δ.

Subcase 2.2. When the function g is unbounded, then functions $\widetilde{f}$ and $\widetilde{g}$ do not exist.

As we have seen in the above considerations, there are cases, when equation (4) is stable and there are cases, when it is not stable.

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