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ON LOCAL WHITNEY CONVERGENCE

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Abstract

In this paper we will give definitions of local Whitney convergence in $\mathcal{F}(X,Y)$ and in $C(X,Y)$, where X is a topological space, (Y,d) is a metric space and $\mathcal{F}(X,Y)$ is the space of all functions from X to Y and $C(X,Y)$ is the space of all continuous functions from X to Y. We will study some properties of this notion and connections with other kinds of convergence.

1. Preliminaries

Throughout the article $(X,\mathcal{T})$ will denote a T_1 topological space and (Y,d) will denote a metric space. For any subset A of the space X, its closure and interior will be denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. Furthermore, $\mathcal{F}(X,Y)$ and $C(X,Y)$ will denote the class of functions and the class of continuous functions from X to Y. Symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the set of real numbers, positive real numbers and positive integers, respectively. If $f\colon X\to Y$, $A\subset X$, then by $f|_A$ we will denote the restriction of f to A.

Definition 1 ([1,2,4,5,6])**.** *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{F}(X,Y)$ is said to be convergent to a function $f\in\mathcal{F}(X,Y)$ in the sense of Whitney, shortly W-convergent, if for each $\varphi\in C(X,\mathbb{R}^+)$ there exists $n_0\in\mathbb{N}$ such that $d\big(f_n(x),f(x)\big)<\varphi(x)$ for each $x\in X$ and $n\geq n_0$.*

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Remark 1. It is obvious that if $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions which is W-convergent to f then f is continuous too.

Since each positive constant function from X to $\mathbb{R}$ belongs to $C(X,\mathbb{R}^+)$, W-convergence implies uniform convergence. It is well-known that if X is a pseudo-compact completely regular topological space then W-convergence and uniform convergence are equivalent. More precisely, we have the following theorem.

Theorem 1 ([1,4])**.** *Let X be a pseudo-compact completely regular topological space and let Y be a metrizable space. The sequence $(f_n)_{n\in\mathbb{N}}$ of functions from X to Y is W-convergent to $f\colon X\to Y$ if and only if it is uniformly convergent to f.*

2. Local Whitney convergence

Now, we give the first definition of local Whitney convergence.

Definition 2. *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{F}(X,Y)$ is said to be convergent to a function $f\in\mathcal{F}(X,Y)$ in the sense of Whitney at a point $x_0\in X$, shortly W-convergent at x_0, if there exists a neighborhood U of x_0 such that for each $\varphi\in C(X,\mathbb{R}^+)$ there exists $n_0\in\mathbb{N}$ such that*

$$d\big(f_n(x),f(x)\big)<\varphi(x)$$

for each $x\in U$ and $n\geq n_0$.

We say that $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent to f if it is W-convergent to f at each $x\in X$.

Corollary 1. *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions which is locally W-convergent to f then f is continuous function too.*

It follows directly from the definitions, that W-convergence implies local W-convergence. Later, we will give an example of locally W-convergent sequence of continuous functions which is not W-convergent.

Theorem 2. *Let $f_n\in\mathcal{F}(X,Y)$ for $n\in\mathbb{N}$ and let $f\in\mathcal{F}(X,Y)$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent to f then $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be locally W-convergent to f. Take any $\varepsilon > 0$ and any compact subset A of X. Let $\varphi \in C(X, \mathbb{R}^+)$ be a constant function, $\varphi(x) = \varepsilon$ for $x \in X$.

Then for each $x \in A$ we can find a neighborhood U_x of x and $n_x \in \mathbb{N}$ such that

$$d\big(f_n(y), f(y)\big) < \varphi(y) = \varepsilon$$

for $y \in U_x$ and $n \geq n_x$.

Since A is compact, $A \subset U_{x_1} \cup \ldots \cup U_{x_k}$ for some $x_1, \ldots, x_k \in A$.

Let $n_0 = \max\{n_{x_1}, \ldots, n_{x_k}\}$. Then $d\left(f_n(x), f(x)\right) < \varepsilon$ for $x \in A$ and $n \geq n_0$. This completes the proof. □

Theorem 3. *Let $(X, \mathcal{T})$ be a paracompact space. The following conditions are equivalent:*

1. *for each metric space (Z, ϱ), every sequence $(f_n)_{n\in\mathbb{N}}$ from $\mathcal{F}(X, Z)$ is locally W-convergent to $f \in \mathcal{F}(X, Z)$ if and only if $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f,*
2. *every sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{F}(X, \mathbb{R})$ is locally W-convergent to $f \in \mathcal{F}(X, \mathbb{R})$ if and only if $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f,*
3. *for each metric space (Z, ϱ) every sequence $(f_n)_{n\in\mathbb{N}}$ from $C(X, Z)$ is locally W-convergent to $f \in C(X, Z)$ if and only if $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f,*
4. *every sequence $(f_n)_{n\in\mathbb{N}}$ from $C(X, \mathbb{R})$ is locally W-convergent to f from $C(X, \mathbb{R})$ if and only if $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f,*
5. *X is locally compact.*

Proof. The implications (1) ⇒ (2), (3) ⇒ (4), (1) ⇒ (3) and (2) ⇒ (4) are obvious.

(4) ⇒ (5) Let local W-convergence and almost uniform convergence in $C(X, \mathbb{R})$ be equivalent. Suppose that X is not locally compact at some $x_0 \in X$. Let $f_n(x) = \frac{1}{n}$, $f(x) = 0$ for $n \in \mathbb{N}$, $x \in X$.

Obviously, the sequence $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f (actually, $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f). Let U be any neighborhood of x_0. Then $\mathrm{cl}(U)$ is not compact and since X is paracompact, it is not pseudo-compact.

Thus there exists $\widetilde{\varphi} \in C(\mathrm{cl}(U), \mathbb{R}^+)$ such that $\inf_{x \in \mathrm{cl}(U)} \widetilde{\varphi}(x) = 0$. Hence $\inf_{x \in U} \widetilde{\varphi}(x) = 0$. By normality of X (X is normal, since it is paracompact), there exists $\varphi \in C(X, \mathbb{R}^+)$ such that $\varphi|_{\mathrm{cl}(U)} = \widetilde{\varphi}$. Then for each $n \in \mathbb{N}$ we have $\left|f_n(x) - f(x)\right| = \frac{1}{n} \geq \varphi(x)$ for some $x \in U$.

Hence the sequence $(f_n)_{n\in\mathbb{N}}$ is not locally W-convergent to f. This contradicts to assumptions. Thus X is locally compact.

(5) $\Rightarrow$ (1) Let X be locally compact and let (Z, ϱ) be any metric space. Take any sequence $(f_n)_{n\in\mathbb{N}}$ from $\mathcal{F}(X, Z)$ and $f \in \mathcal{F}(X, Z)$. If the sequence is locally W-convergent to f then, by Theorem 2, it is almost uniformly convergent to f. Assume that $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f. Let $x_0 \in X$. By local compactness of X there exists a neighborhood U of x_0 such that $\mathrm{cl}(U)$ is compact. Then $(f_n|_{\mathrm{cl}(U)})_{n\in\mathbb{N}}$ is uniformly convergent to $f|_{\mathrm{cl}(U)}$. Let $\varphi \in C(X, \mathbb{R}^+)$. Then $\inf_{x \in \mathrm{cl}(U)} \varphi(x) = c > 0$ and there exists $n_0 \in \mathbb{N}$ such that $\varrho\big(f_n(x), f(x)\big) < c \leq \varphi(x)$ for $x \in \mathrm{cl}(U)$ and $n \geq n_0$. It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent to f. The proof is completed. □

Theorem 4. *Let $(X, \mathcal{T})$ be a normal space and let $f_n \in \mathcal{F}(X, Y)$ for $n \in \mathbb{N}$ and $f \in \mathcal{F}(X, Y)$. For each $x_0 \in X$ the following conditions are equivalent:*

1. *the sequence $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f at x_0,*
2. *there exists a neighborhood U of x_0 such that the sequence $\big(f_n|_{\mathrm{cl}(U)}\big)_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(U)}$,*
3. *for each neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that $\big(f_n|_{\mathrm{cl}(V)}\big)_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(V)}$,*
4. *there exists a neighborhood U of x_0 such that for each neighborhood $V \subset U$ of x the sequence $\big(f_n|_{\mathrm{cl}(V)}\big)_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(V)}$.*

Proof. 2) $\Rightarrow$ 1) Let U be a neighborhood of a point x_0 such that the sequence $\big(f_n|_{\mathrm{cl}(U)}\big)_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(U)}$. Take any $\varphi \in C(X, \mathbb{R}^+)$. Then $\varphi_{|\,\mathrm{cl}(U)} \in C(\mathrm{cl}(U), \mathbb{R}^+)$. Therefore there exists $n_0 \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varphi_{|\,\mathrm{cl}(U)}(x)$$

for each $x \in \mathrm{cl}(U)$ and $n \geq n_0$. In particular,

$$d(f_n(x), f(x)) < \varphi(x)$$

for each $x \in U$ and $n \geq n_0$. It follows that $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f at x_0.

3) $\Rightarrow$ 2) This implication is evident.

4) $\Rightarrow$ 3) This implication is evident too.

1) $\Rightarrow$ 4) Let the sequence $(f_n)_{n\in\mathbb{N}}$ be W-convergent to f at x_0. There exists a neighborhood U of x_0 such that for $\varphi \in C(X, \mathbb{R}^+)$ we can find $n_0 \in \mathbb{N}$ for which $d(f_n(x), f(x)) < \varphi(x)$ for $n \geq n_0$ and $x \in U$. Since X is normal, there exists a neighborhood U_1 of x_0 such that $\mathrm{cl}(U_1) \subset U$. Let V be any neighborhood of x_0 contained in U_1. Then $\mathrm{cl}(V) \subset \mathrm{cl}(U_1) \subset U$.

We claim that the sequence $(f_n|_{\mathrm{cl}(V)})_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(V)}$. Let $\widetilde{\varphi} \in C(\mathrm{cl}(V), \mathbb{R}^+)$. Since X is normal, there exists $\varphi \in C(X, \mathbb{R}^+)$ such that $\varphi|_{\mathrm{cl}(V)} = \widetilde{\varphi}$. Then we can find $n_0 \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varphi(x) = \widetilde{\varphi}(x)$$

for $x \in U$ and $n \geq n_0$. In particular, $d(f_n(x), f(x)) < \varphi(x) = \widetilde{\varphi}(x)$ for $x \in \mathrm{cl}(V)$ and $n \geq n_0$. Thus $\left(f_n|_{\mathrm{cl}(V)}\right)_{n\in\mathbb{N}}$ is W-convergent to $f|_{\mathrm{cl}(V)}$ and the proof is completed. $\square$

We will need the following result.

Theorem 5 ([3, Theorem 4])**.** *Let (X, τ) be a normal topological space, (Y, d) a metric space. Let $f \in C(X, Y)$ and $f_n \in C(X, Y)$ for $n \in \mathbb{N}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f if and only if the following two conditions hold:*

1. *the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f;*
2. *there exists a closed countably compact set $K \subset X$ such that if U is an open subset of X containing K then we can find $n_0 \in \mathbb{N}$ such that $f_n|_{(X\setminus U)} = f|_{(X\setminus U)}$ for $n \geq n_0$.*

Problem 1. *Is Theorem 5 true for functions from $\mathcal{F}(X, Y)$?*

Theorem 6. *Let X be a paracompact space. Assume that $f_n \in C(X, Y)$ for $n \in \mathbb{N}$ and $f \in C(X, Y)$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent to f if and only if it is almost uniformly convergent and there exist a locally compact set $F \subset X$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of open subsets of X such that*

1. *$f_k(x) = f(x)$ for all $x \in A_n$ and $k \geq n$,*

(2) $X \setminus F \subset \bigcup_{n=1}^{\infty} A_n$,

(3) *for each* $x \in F$ *there exists a neighborhood* G_x *of* x *such that for any neighborhood* V *of* $F \cap G_x$ *there exists* n_0 *such that* $f_n(y) = f(y)$ *for all* $y \in G_x \setminus V$ *and* $n \geq n_0$.

Proof. First assume that $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent to f. Obviously, $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent. By Theorem 4, for each $x \in X$ we can find a neighborhood $\widetilde{U}_x$ of x such that $\left(f_n|_{\mathrm{cl}(\widetilde{U}_x)}\right)_{n\in\mathbb{N}}$ is Whitney convergent to $f|_{\mathrm{cl}(\widetilde{U}_x)}$. By Theorem 4 and by normality of X, there exists a neighborhood U_x of x such that $\mathrm{cl}(U_x) \subset \widetilde{U}_x$ and $\left(f_n|_{\mathrm{cl}(U_x)}\right)_{n\in\mathbb{N}}$ is Whitney convergent to $f|_{\mathrm{cl}(U_x)}$.

Let $F_x \subset \mathrm{cl}(\widetilde{U}_x)$ be a closed countably compact set which satisfies conditions of Theorem 5 for the sequence $(f_n|_{\mathrm{cl}(\widetilde{U}_x)})_{n\in\mathbb{N}}$. By paracompactness of X, F_x is locally compact.

Take any $y \in \mathrm{cl}(U_x) \setminus F_x \subset \widetilde{U}_x \setminus F_x$. By normality of X, there exist a neighborhood V of y such that $\mathrm{cl}(V) \cap F_x = \emptyset$ and $\mathrm{cl}(V) \subset \widetilde{U}_x$. Then, by Theorem 5, we can find $n_0 \in \mathbb{N}$ such that $f_n(t) = f(t)$ for each $t \in V$ and each $n \geq n_0$. Thus we have proven that

(*) for each $y \in \mathrm{cl}(U_x) \setminus F_x$ there exist a neighborhood V of y and $n_0 \in \mathbb{N}$ such that $f_n(t) = f(t)$ for each $t \in V$ and each $n \geq n_0$.

Put

$$A_n = \mathrm{int}\{x \in X \colon f_k(x) = f(x) \text{ for each } k \geq n\}$$

for $n \in \mathbb{N}$ and

$$F = X \setminus \bigcup_{n=1}^{\infty} A_n.$$

Then F is closed and, by $(*)$, $F \cap \mathrm{cl}(U_x) \subset F_x$ for $x \in X$. Hence F is locally compact. Take any $x \in F$ and put $G_x = U_x$. Then condition (3) is satisfied, by Theorem 5 and by definition of U_x.

Now, assume that $(f_n)_{n\in\mathbb{N}}$ is almost uniformly convergent to f and there exist a locally compact set $F \subset X$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of open subsets of X for which conditions (1), (2) and (3) hold. Take any $x_0 \in X$. First consider the case, where $x_0 \notin F$.

Then $x_0 \in A_k$ for some $k \in \mathbb{N}$. Since X is normal, there exists a neighborhood U of x_0 such that $\mathrm{cl}(U) \subset A_k$. Then $f_n|_{\mathrm{cl}(U)} = f|_{\mathrm{cl}(U)}$ for $n \geq k$. Hence

$(f_n|\mathrm{cl}(U))_{n\in\mathbb{N}}$ is Whitney convergent to $f|_{\mathrm{cl}(U)}$. By Theorem 4, $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f at x_0.

Finally, consider the case, where $x_0 \in F$. Let G_{x_0} be a neighborhood of x_0 such that for any neighborhood V of $F \cap G_{x_0}$ there exists n_0 such that $f_n(x) = f(x)$ for all $x \in G_{x_0} \setminus V$ and $n \geq n_0$.

Since F is locally compact and X is normal, there exists a neighborhood W_{x_0} of x_0 such that $\mathrm{cl}(W_{x_0}) \cap F$ is compact and $\mathrm{cl}(W_{x_0}) \subset G_{x_0}$. Consider the sequence $(f_n|_{\mathrm{cl}(W_{x_0})})_{n\in\mathbb{N}}$. Clearly, it is uniformly convergent to $f|_{\mathrm{cl}(W_{x_0})}$ and $F \cap \mathrm{cl}(W_{x_0})$ is compact. Let V be any neighborhood of $F \cap \mathrm{cl}(W_{x_0})$ in $\mathrm{cl}(W_{x_0})$. There exists open subset $\widetilde{V}$ of X such that $V = \widetilde{V} \cap \mathrm{cl}(W_{x_0})$.

Then $(G_{x_0} \setminus \mathrm{cl}(W_{x_0})) \cup \widetilde{V})$ is a neighborhood of $F \cap G_{x_0}$. Therefore we can find n_0 such that $f_n|_{\mathrm{cl}(W_{x_0})}(t) = f|_{\mathrm{cl}(W_{x_0})}(t)$ for each $t \in \mathrm{cl}(W_{x_0}) \setminus V$. By Theorem 5, $(f_n|_{\mathrm{cl}(W_{x_0})})_{n\in\mathbb{N}}$ is Whitney convergent to $f|_{\mathrm{cl}(W_{x_0})}$ and, by Theorem 4, $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f at x_0.

Since x_0 was arbitrary, the proof is completed. □

3. Strong local W-convergence and $w^\star$-convergence

Theorem 4 motivates us to introduce two new definitions of local Whitney convergence.

Definition 3. *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{F}(X,Y)$ is said to be $w^\star$-convergent to $f \in \mathcal{F}(X,Y)$ at a point $x_0 \in X$, if there exists a neighborhood U of x_0 such that for each $\varphi \in C(U,\mathbb{R}^+)$ there exists $n_0 \in \mathbb{N}$ such that*

$$d\left(f_n(x), f(x)\right) < \varphi(x)$$

for each $x \in U$ and $n \geq n_0$.

Equivalently, $(f_n)_{n\in\mathbb{N}}$ is $w^\star$-convergent to a function $f \in \mathcal{F}(X,Y)$ at a point $x_0 \in X$, if there exists a neighborhood U of the point x_0 such that $(f_n|_U)_{n\in\mathbb{N}}$ is W-convergent to $f|_U$.

We say that $(f_n)_{n\in\mathbb{N}}$ is locally $w^\star$-convergent to f if it is $w^\star$-convergent to f at every $x \in X$.

Definition 4. *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\mathcal{F}(X,Y)$ is said to be W-convergent in the strong sense to $f \in \mathcal{F}(X,Y)$ at a point $x_0 \in X$, shortly SW-convergent at x_0, if for each neighborhood U of x_0 we can find* *a neighborhood* $V \subset U$ *of* x_0 *such that for each* $\varphi \in \mathcal{C}(V, \mathbb{R}^+)$ *there exists* $n_0 \in \mathbb{N}$ *such that*

$$d\big(f_n(x), f(x)\big) < \varphi(x)$$

for all $x \in V$ *and* $n \geq n_0$.

Equivalently, $(f_n)_{n\in\mathbb{N}}$ *is SW-convergent to a function* $f \in \mathcal{F}(X,Y)$ *at a point* $x_0 \in X$*, if for each neighborhood* U *of* x_0 *there exists a neighborhood* $V \subset U$ *of* x_0 *such that the sequence* $(f_n|_V)_{n\in\mathbb{N}}$ *is* W*-convergent to* $f|_V$.
We say that $(f_n)_{n\in\mathbb{N}}$ *is locally SW-convergent to* f *if it is SW-convergent to* f *at every* $x \in X$.

Corollary 2. *If a sequence* $(f_n)_{n\in\mathbb{N}}$ *of continuous functions from* X *to* Y *is locally SW-convergent or locally* $w^\star$*-convergent to* f *then* $f\colon X \to Y$ *is continuous too.*

From the definitions we easily get the following two propositions.

Proposition 1. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence from* $\mathcal{F}(X,Y)$, $f \in \mathcal{F}(X,Y)$ *and* $x_0 \in X$.

(1) *If* $(f_n)_{n\in\mathbb{N}}$ *is* $w^\star$*-convergent to* f *at* x_0 *then* $(f_n)_{n\in\mathbb{N}}$ *is* W*-convergent to* f *at* x_0.
(2) *If* $(f_n)_{n\in\mathbb{N}}$ *is SW-convergent to* f *at* x_0 *then* $(f_n)_{n\in\mathbb{N}}$ *is* $w^\star$*-convergent to* f *at* x_0.

The first relation follows from the obvious fact that if $\varphi \in C(X, \mathbb{R}^+)$ then $\varphi|_U \in C(U, \mathbb{R}^+)$.

Proposition 2. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions from* $\mathcal{F}(X,Y)$ *and* $f \in \mathcal{F}(X,Y)$*. If* $(f_n)_{n\in\mathbb{N}}$ *is* W*-convergent to function* f *then* $(f_n)_{n\in\mathbb{N}}$ *is locally* $w^\star$*-convergent to* f.

We can put $U = X$ in the definition of $w^\star$-convergence.

Relationships between discussed types of convergence can be illustrated in the following diagrams.

$$\textbf{SW-conv. at } x_0 \;\Rightarrow\; w^\star\textbf{-conv. at } x_0 \;\Rightarrow\; \textbf{W-conv. at } x_0$$

$$\begin{array}{ccccc} & & \textbf{\textit{W}-convergence} & & \\ & & \Downarrow & & \\ \textbf{local SW-conv.} & \Rightarrow & \textbf{local } w^\star\textbf{-conv.} & \Rightarrow & \textbf{local W-conv.} \end{array}$$

We will show that none of the reverse implications hold, even in $C(X, Y)$.

Example 1. Let $f_n \colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0 & if \quad x \leq n, \\ \frac{1}{n} & if \quad x \geq n+1, \\ \frac{x-n}{n} & if \quad n < x < n+1, \end{cases}$$

for $n \in \mathbb{N}$ and let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ if $x \in \mathbb{R}$.

Then f and f_n for $n \in \mathbb{N}$ are continuous. Since for each $x_0 \in \mathbb{R}$ there exist $n_0 \in \mathbb{N}$ such that $f_n(x) = 0$ if $n \geq n_0$ and $x \in (x_0 - 1, x_0 + 1)$, the sequence $(f_n)_{n\in\mathbb{N}}$ is locally SW-convergent to f. On the other hand, there exists $\varphi \in C(\mathbb{R}, \mathbb{R}^+)$ such that $\varphi(n+1) = \frac{1}{n}$ for $n \geq 1$. Then $|f_n(n+1) - f(n+1)| = \frac{1}{n} = \varphi(n+1)$ for each n. It follows that $(f_n)_{n\in\mathbb{N}}$ is not W-convergent to f.

Example 2. Let $f_n \colon [0,1] \to \mathbb{R}$, $f_n(x) = \frac{1}{n}$ if $x \in [0,1]$, $n \in \mathbb{N}$ and let $f \colon [0,1] \to \mathbb{R}$, $f(x) = 0$ if $x \in [0,1]$. Then functions f and f_n are continuous. Since $[0,1]$ is compact and $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f, $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f.

Take any $x \in (0,1)$ and let $U = (0,1)$. Then U is a neighborhood of x and any neighborhood V of x contained in U is not closed. Therefore there exist $\varphi \in C(V, \mathbb{R}^+)$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset V$ such that $\varphi(x_n) = \frac{1}{n}$. Hence $(f_n|_V)_{n\in\mathbb{N}}$ is not W-convergent to $f|_V$. Thus $(f_n)_{n\in\mathbb{N}}$ is not SW-convergent to f at any $x \in (0,1)$. Hence $(f_n)_{n\in\mathbb{N}}$ is not locally SW-convergent to f.

Example 3. Let $f_n \colon \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}$ if $n \in \mathbb{N}$, $x \in \mathbb{R}$ and let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ if $x \in \mathbb{R}$. Obviously, $(f_n)_{n\in\mathbb{N}}$ is not $w^\star$-convergent to f at any point $x \in \mathbb{R}$ (arguments are similar as in the previous example).

Let $x_0 \in \mathbb{R}$ and $\varphi \in C(\mathbb{R}, \mathbb{R}^+)$. Then there exists $\delta > 0$ such that $\varphi(x) \geq \delta$ for $x \in (x_0 - 1, x_0 + 1)$. Hence $|f_n(x) - f(x)| = \frac{1}{n} < \delta \leq \varphi(x)$ for $x \in (x_0 - 1, x_0 + 1)$ and sufficiently large n. Thus $(f_n)_{n\in\mathbb{N}}$ is locally W-convergent.

The next example shows that $w^\star$-convergence at a point is not a local property.

Example 4. Let $f_n, g_n\colon [0,2] \to \mathbb{R}$, $f_n(x) = \frac{1}{n}$ if $x \in [0,2]$ and

$$g_n(x) = \begin{cases} \frac{1}{n} & if \quad x \in \{0\} \cup \left[\frac{1}{n}, 2\right], \\ 1 & if \quad x = \frac{1}{2n}, \\ \text{linear in } \left[0, \frac{1}{2n}\right] \text{ and } \left[\frac{1}{2n}, \frac{1}{n}\right], \end{cases}$$

if $n \in \mathbb{N}$. Next, let $f\colon [0,2] \to \mathbb{R}$, $f(x) = 0$ if $x \in [0,2]$. Since $[0,2]$ is compact and $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f, $(f_n)_{n\in\mathbb{N}}$ is locally $w^\star$-convergent to f.

Take any $x \in [0,2]$. Let U be any neighborhood of x. Then either $[0,\eta) \subset U$ for some $\eta > 0$ or U is not closed subset of $[0,2]$.

If $[0,\eta) \subset U$ for some $\eta > 0$ then $(g_n|_U)_{n\in\mathbb{N}}$ is not uniformly convergent. Hence $(g_n|_U)_{n\in\mathbb{N}}$ is not Whitney convergent.

Finally, assume that U is not a closed subset of $[0,2]$. Then there exist a sequence $(x_n)_{n\in\mathbb{N}} \subset U$ and $\varphi \in C(U, \mathbb{R}^+)$ such that $\varphi(x_n) = \frac{1}{n}$ for $n \geq 1$. Since $g(x) \geq \frac{1}{n}$ for $x \in [0,2]$, we have

$$|g_n(x_n) - f(x_n)| \geq \frac{1}{n} = \varphi(x_n).$$

It follows that $(g_n|_U)_{n\in\mathbb{N}}$ is not Whitney convergent. We have proven that $(g_n)_{n\in\mathbb{N}}$ is not $w^\star$-convergent at any $x \in [0,2]$. Thus the sequence $(f_n)_{n\in\mathbb{N}}$ is $w^\star$-convergent at each point $x \in [0,2]$ and $(g_n)_{n\in\mathbb{N}}$ is not $w^\star$-convergent at any $x \in [0,2]$, nevertheless $f_n(x) = g_n(x)$ for all $x \in [1,2]$ and $n \in \mathbb{N}$.

Theorem 7. *Let $(X, \mathcal{T})$ be a paracompact perfect topological space, $x_0 \in X$ and let (Y, d) be any metric space. If X is locally compact at x_0 then the following conditions are equivalent:*

1. *W-convergence at x_0 and SW-convergence at x_0 in $\mathcal{F}(X,Y)$ are equivalent.*
2. *W-convergence at x_0 and SW-convergence at x_0 in $C(X,Y)$ are equivalent.*
3. *there exists a local base of $\mathcal{T}$ at x_0 consisting of sets which are both open and closed.*

Proof. 3) $\Rightarrow$ 1) Assume that there exists a local base at x_0 consisting of sets which are both open and closed. Let $(f_n)_{n\in\mathbb{N}}$ be any sequence from $\mathcal{F}(X,Y)$ and $f \in \mathcal{F}(X,Y)$. If $(f_n)_{n\in\mathbb{N}}$ is SW-convergent to f at x_0 then, obviously, it is W-convergent at x_0.

On the other hand, assume that $(f_n)_{n\in\mathbb{N}}$ is W-convergent to f at x_0 and let U be any neighborhood of x_0. By Theorem 4, there exists a neighborhood U_1 of x_0 such that for each neighborhood U_2 of x_0 contained in U_1 the sequence $(f_n|_{U_2})_{n\in\mathbb{N}}$ is W-convergent to $f|_{U_2}$. By assumption, we can find a neighborhood V of x_0 such that $V \subset U_1 \cap U$ and $\mathrm{cl}(V) = V$. Then $(f_n|_V)_{n\in\mathbb{N}}$ is W-convergent to $f|_V$. It follows that $(f_n)_{n\in\mathbb{N}}$ is SW-convergent to f at x_0. Thus W-convergence at x_0 and SW-convergence at x_0 in $\mathcal{F}(X,Y)$ are equivalent.

1) $\Rightarrow$ 2) This implication is obvious.

2) $\Rightarrow$ 3) Assume that W-convergence at x_0 and SW-convergence at x_0 are equivalent in $C(X,Y)$. Suppose that there exists a neighborhood U of x_0 such that each neighborhood V of x_0 contained in U is not closed. By local compactness of X at x_0, there exists a neighborhood U_0 of x_0 with compact closure. Let $f\colon X \to Y$, $f(x) = 0$ for $x \in X$. By normality of X, we can find a neighborhood G of x_0 and a sequence of continuous functions $f_n\colon X \to Y$ such that $\mathrm{cl}(G) \subset U \cap U_0$, $f_n(x) = \frac{1}{n}$ for $x \in G$, $f_n(x) = 0$ for $x \in X \setminus (U \cap U_0)$ and $0 \le f_n(x) \le \frac{1}{n}$ for $x \in X$. By Theorem 5, $(f_n)_{n\in\mathbb{N}}$ is Whitney convergent to f, because $\mathrm{cl}(U \cap U_0)$ is compact. Hence $(f_n)_{n\in\mathbb{N}}$ is W-convergent at x_0.

Let $V \subset G$ be any neighborhood of x_0. In particular, $V \subset U$. Therefore V is not closed. It follows that V is not compact. Since X is perfect, V is of type F_σ. Therefore V is a paracompact space. It follows that V is non pseudo-compact. Hence there exist a sequence $(x_n)_{n\in\mathbb{N}}$ of points from V and $\varphi \in C(V,\mathbb{R}^+)$ such that $\varphi(x_n) = \frac{1}{n}$ for $n \ge 1$. Then

$$d\big(f_n(x_n), f(x_n)\big) = \tfrac{1}{n} = \varphi(x_n)$$

for all n. It follows that $(f_n|_V)_{n\in\mathbb{N}}$ is not Whitney convergent to $f|_V$ and therefore $(f_n)_{n\in\mathbb{N}}$ is not SW-convergent to f at x_0, which is a contradiction. This completes the proof. □

Since every metric space is paracompact and perfect we have the following corollary.

Corollary 3. *Let (X, d_X) be a locally compact metric space. The following conditions are equivalent:*

(1) *for each metric space (Z, ϱ) local W-convergence and local SW-convergence in $\mathcal{F}(X, Z)$ are equivalent.*

(2) *local W-convergence and local SW-convergence in $\mathcal{F}(X, \mathbb{R})$ are equivalent.*

(3) *for each metric space (Z, ϱ) local W-convergence and local SW-convergence in $C(X, Z)$ are equivalent.*

(4) *local W-convergence and local SW-convergence in $C(X, \mathbb{R})$ are equivalent.*

(5) *X is zero dimensional.*

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