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SOME PROPERTIES OF i–CONNECTED SETS (PART II)

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Abstract. A generalization theorem for i–connected sets in the Hashimoto topology is given. Moreover, i–connectivity in the topology of at most countable complements and in the order topology is presented.

1. Introduction

Let (X, T) be a topological space and let P stand for some property of the subsets of X. We denote by $\mathcal{P}$ the family of all subsets of X which satisfy P. We say that a subset A has the property P at the point $p \in X$, if there exists a neighbourhood V_p of p such that $V_p \cap A \in \mathcal{P}$. We introduce the symbol A^* to define the set of all points at which A does not have the property P, i.e.

$$A^* = \left\{ p \in X : \bigwedge_{V_p} (V_p \cap A \notin \mathcal{P}) \right\}. \tag{1}$$

Additionally, let us assume that the family $\mathcal{P}$ is ideal, i.e.

(2) the relations $A \in \mathcal{P}$ and $B \in \mathcal{P}$ imply $A \cup B \in \mathcal{P}$,

(3) the relations $A \in \mathcal{P}$ and $B \subset A$ imply $B \in \mathcal{P}$.

Moreover, let us assume that the property P satisfies the following

(4) $(A \in \mathcal{P}) \Leftrightarrow (A \cap A^* = \phi) \Leftrightarrow (A^* = \phi)$,

and that

(5) every single element subset of X belongs to $\mathcal{P}$.

Example 1. The family of the sets of the first category and the family of the sets of measure zero in the sense of Lebesgue satisfy conditions (2)-(5).

Let (X, T) be T_1 – space. Using an ideal $\mathcal{P}$ we can introduce the new topology on the set X, the so–called Hashimoto topology, defined by the formula

$$T^* = \{U \backslash F \subset X : U \in T \wedge F \in \mathcal{P}\}. \tag{6}$$

We see easily that $T \subset T^*$.

For simplicity of the notation we continue to write $\mathrm{int}A\,(\mathrm{cl}A)$ for the interior (the closure) of A in the topological space (X, T) and $\mathrm{int}_*A\,(\mathrm{cl}_*A)$ for the interior (the closure) of A in the space (X, T^*).

Because T^* is a stronger topology we have

(7) $\bigwedge_{A \subset X} \mathrm{int}A \subset \mathrm{int}_*A,$

(8) $\bigwedge_{A \subset X} \mathrm{cl}_*A \subset \mathrm{cl}A,$

(9) if $M \subset X$ is connected in the Hashimoto topology (X, T^*), then M is connected in (X, T).

Now let us recall the definition of an i–connected set (cf. [2]).

Definition 1. Let (X, T) be a topological space. A set $A \subset X$ is said to be *i–connected* if it has a nonempty interior and both A and $\mathrm{int}A$ are connected.

Example 2. In the natural topology on the straight line every connected set which has a nonempty interior is i–connected. Note that no similar fact holds for the Euclidean plane. For instance, a set consisting of two tangent discs is connected but it's interior is not.

2. The i–connected sets in the Hashimoto topology

We will need the following lemmas.

Lemma 1. (cf. [1], p. 6). Let (X, T) be a topological space. An open set G belonging to $\mathcal{P}$ is contained in $X \backslash X^*$.

Lemma 2. (cf. [1], p. 7). If the space (X, T) satisfies $X^* = X$, then $\mathrm{cl}G = \mathrm{cl}_*G$ for every $G \in T^*$.

Lemma 3. Let (X, T) be a topological space and let $X^* = X$. If $A \subset X$ is open and connected in (X, T), then A is connected in (X, T^*).

Proof. Let us suppose that A is disconnected in (X,T^*). Then we can represent the set A as the sum of two nonempty disjoint open subsets in the subspace (A,T^*_A). Since $A\in T$ and $T\subset T^*$, (6) shows that there exist two nonempty sets $U_1,U_2\in T$ and two sets $F_1,F_2\in\mathcal{P}$ such that $A=(U_1\backslash F_1)\cup(U_2\backslash F_2)$ and $(U_1\backslash F_1)\cap(U_2\backslash F_2)=\phi$. Hence $A\subset U_1\cup U_2$ and

$$U_1\cap U_2\subset F_1\cup F_2. \tag{10}$$

Because $U_1\cap U_2\in T, F_1\cup F_2\in\mathcal{P}$ and $X=X^*$, therefore, by Lemma 1 and (10), we obtain that $U_1\cap U_2=\phi$ which means that A is disconnected in (X,T), contrary to the assumption. □

Theorem 1. Let (X,T) be a topological space and let $X=X^*$. If a set $A\subset X$ has a nonempty and connected interior in the space (X,T) and $A\subset \mathrm{cl}\,\mathrm{int}A$, then A is i–connected in the space (X,T^*).

Proof. Since int$A$ in open and connected in (X,T), by Lemma 3, it is connected in (X,T^*). By (7) we have

$$\mathrm{int}A\subset\mathrm{int}_*A\subset A\subset\mathrm{cl}A.$$

Moreover, by the assumption and Lemma 2, we get

$$\mathrm{cl}A=\mathrm{cl}\,\mathrm{int}A=\mathrm{cl}_*\,\mathrm{int}A$$

and finally,

$$\mathrm{int}A\subset\mathrm{int}_*A\subset A\subset\mathrm{cl}_*\,\mathrm{int}A.$$

Since int$A$ and $\mathrm{cl}_*\,\mathrm{int}A$ (as the closure of a connected set) are connected in (X,T^*), the sets int_*A and A are connected in (X,T^*), and the proof is completed. □

Note that every set satisfying the assumptions of the above theorem is i–connected in (X,T). Therefore we obtain the following

Corollary 1. If a set $A\subset X$ is i–connected in the space (X,T) and $X=X^*$ and $\mathrm{cl}A=\mathrm{cl}\,\mathrm{int}A$, then A is i–connected in the space (X,T^*).

Remark 1. Taking in the above theorem $(X,T)=(\mathbb{R}^2,T_d)$ and $T^*=\{U\backslash F\subset\mathbb{R}^2 : U\in T_d \text{ and } \mu(F)=0\}$, where μ denotes the Lebesgue measure in $\mathbb{R}^2$ and T_d is the family of open sets in the Euclidean plane, one gets Theorem 3 of [2].

3. The i–connectivity in the topology of at most countable complements

We start with the following lemma which gives the equivalent condition on connectivity of open sets.

Lemma 4. Let (X, T) be a topological space and let a nonempty and open set $A \subset X$ be fixed. Then a set A is connected in the space (X, T) if and only if the following condition is fulfilled

$$\bigwedge_{A_1 \subset A} [(A_1 \neq \phi \wedge A_1 \neq A \wedge A_1 \in T) \Rightarrow A \backslash A_1 \notin T]. \tag{11}$$

Proof.

$\Rightarrow$ Let us suppose, contrary to our claim, that there exists a nonempty and open set A_1 such that $A_1 \subset A$, $A_1 \neq A$ and $A \backslash A_1 \in T$. Then the set A can be represented as the union of two nonempty disjoint and open sets A_1 and $A \backslash A_1$, which is impossible.

$\Leftarrow$ Conversely, suppose that there exist two nonempty open sets $A_1, A_2 \subset X$ such that $A \cap A_1 \neq \emptyset, A \cap A_2 \neq \emptyset$, $(A \cap A_1) \cap (A \cap A_2) = \emptyset$ and $A = (A \cap A_1) \cup (A \cap A_2)$. Since A is open therefore $A \cap A_1 \in T$ and $A \cap A_2 \in T$. It follows that there exists a nonempty and open set $B = A \cap A_1$ such that $B \subset A, B \neq A$ and $A \backslash B \in T$, contrary to (11). $\square$

Let $\mathfrak{M}$ be a family of all subsets of X which satisfy (11), i.e.

$$\mathfrak{M} := \left\{ A \subset X : \bigwedge_{A_1 \subset A} [(A_1 \neq \phi \wedge A_1 \neq A \wedge A_1 \in T) \Rightarrow A \backslash A_1 \notin T] \right\}.$$

We present some properties of the family $\mathfrak{M}$:

(a) A nonempty and open set is connected if and only if it belongs to $\mathfrak{M}$.

(b) A nonempty and open set is i–connected if and only if it belongs to $\mathfrak{M}$.

(c) If a set is i–connected, then it's interior belongs to $\mathfrak{M}$.

(d) A topological space (X, T) is connected if and only if X belongs to $\mathfrak{M}$.

(e) A topological space (X, T) is connected if and only if every nonempty and closed subset of X belongs to $\mathfrak{M}$.

Remark 2. If we denote by $\mathcal{S}$ the family of all connected sets in the topological space (X, T), then

$$\mathcal{S} \cap (T \backslash \{\emptyset\}) = \mathfrak{M} \cap (T \backslash \{\emptyset\}).$$

Example 3. The base

$$\mathcal{B} = \{(a, b) : a, b \in R,\ a < b\}$$

of natural topology on the straight line is contained in the family $\mathfrak{M}$.

In the sequel the symbol card$F$ denotes the cardinality of the set $F \subset X$.

Theorem 2. Let X be an uncountable set and let T_S be a topology of at most countable complements, i.e.

$$T_s = \left\{ U \subset X : U = \phi \vee \bigvee_{F \subset X} (\mathrm{card} F \leq \chi_0 \wedge U = X \backslash F) \right\}.$$

Then every nonempty and open set belongs to $\mathfrak{M}$.

Proof. Let us choose arbitrary nonempty open sets U and A_1 such that $A_1 \subset U$ and $A_1 \neq U$. Then there exist two at most countable sets $F, F_1 \subset X$ such that $U = X \backslash F$, $A_1 = X \backslash F_1$, $F \subset F_1$ and $F \neq F_1$.
Since

$$U \backslash A_1 = U \backslash (X \backslash F_1) = U \cap (X \backslash (X \backslash F_1)) = U \cap F_1$$

therefore, the set $U \backslash A_1$ is nonempty and at most countable, and finally $U \backslash A_1 \notin T_S$ which completes the proof. □

By the above theorem and Lemma 4 we have the following corollaries:

Corollary 2. Every nonempty and open set in (X, T_S) is connected.

Corollary 3. Every connected set in (X, T_S) which has a nonempty interior, is i–connected.

Therefore in the space (X, T_S) the connectivity of the sets with nonempty interiors is identical with the i–connectivity.

We end our study by the following observation.

Let (X, T) be a topological space such that X is an uncountable set and $T = \{\emptyset, X\}$. Moreover, let $\mathcal{P}$ be a family of subsets of X defined by the formula

$$\mathcal{P} = \{A \subset X : \mathrm{card} A \leq \chi_0\}.$$

It is easy to check that the family $\mathcal{P}$ fulfils conditions (2)-(5) and so we can introduce the Hashimoto topology T^* in the set X. It immediately follows that such defined topology is the same as the topology of at most countable complements.

4. The i–connectivity in the order topology

Let $(X, \leq)$ be linearly ordered set including at least two elements where the ordering relation $\leq$ is dense and does not have gaps (i.e. the relation $\leq$ is continuously ordered X).

Obviously in X we can introduce the so–called order topology $\mathcal{T}$ by defining a base consisting of all open intervals of the form

$$(a, b) = \{x \in X : a < x < b\}, \quad (\leftarrow, a) = \{x \in X : x < a\},$$

$$(a, \rightarrow) = \{x \in X : a < x\}, \quad \text{where } a, b \in X \text{ and } a < b.$$

It follows immediately that in such defined topology every connected set is convex. Moreover, in [3] it has been shown that every convex set is connected. Hence for continuously ordered subsets (in particular on the straight line) the connectivity is equivalent to it's convexity.

Now, let us quote the following theorem.

Theorem 3 (cf. [3], p. 13). Let $(X, \leq)$ be linearly ordered set including at least two elements where the ordering relation $\leq$ deos not have gaps. Then every nonempty and convex subset is one of the form:

$$X, (a, b), (-\infty, a), (a, \rightarrow), [a, b], (\leftarrow, a], [a, \rightarrow), (a, b], [a, b) \tag{12}$$

where $[a, b] = \{x \in X : a \leq x \leq b\}, (\leftarrow, a] = \{x \in X : x \leq a\}, (a, \rightarrow) = \{x \in X : a \leq x\}, (a, b] = \{x \in X : a < x \leq b\}, [a, b) = \{x \in X : a < x \leq b\}$ for arbitrary $a, b \in X$ and $a < b$.

Corollary 4. Let $(X, \mathcal{T})$ be a topological space defined by the ordering relation $\leq$ which is dense and does not have gaps. Then the only nonempty connected sets are subsets of the form (12).

Corollary 5. Let $(X, \mathcal{T})$ be a topological space defined by the ordering relation $\leq$ which is dense and does not have gaps. Then every connected set which has a nonempty interior is i–connected.

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