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Logical independence, its algebraic generalization and applications

Summary

We present a short history of the notion of logical independence, its counterpart and generalization in universal algebra, and their applications in different areas.

Keywords: logical independence, free axiomatization, algebraic independence, free algebras.

Logical independence can refer to sets of formulae, sentences or rules, however, it can be also understood as a relation between a formula and a set of formulae. We say that a formula α is independent from a set S of formulae if α is not provable from S.

The set S of formulae is called independent if each its formula α is independent from the set $S\alpha\}$.

Sometimes instead of independent sets we can discuss countably independent sets. The set S of formulae is countably independent if every one of its countable subset is independent.

As we see, independence in logic is always related to the notion of provability and therefore related to some logic, the notion of consequence or the notion of proof. What is more, in the classical case, independence is strictly related to the notion of consistency, since according to the definition, a formula α is independent from a set S of formulae if and only if the set $S \cup \{\neg\alpha\}$ is consistent.

In general, independence is usually expected from the set of axioms of a given theory, sometimes also from the set of primitive rules of a given formal system. From the theoretical point of view, independence is not an essential property of an axiomatization, since a dependent system of axiom is correct and can serve well. However, in many cases this property is desired, either to reach the conclusion of a reduced set of axioms, or to be able to replace an independent axiom by another one in order to produce a more concise system.

Logical independence was one of four pillars of Hilbert's project. It was meant as a formal counterpart of the intuitive notion of simplicity. The notion of independence has played an important role in the methodology of deductive science, inspiring philosophical discussions and influencing a development of formal logic. Since the beginning of the last century it has also gained the crucial significance in the foundations of physics, in particular in the quantum physics.

The first formal solution to the problem of independent axiomatization was given by Alfred Tarski (see Tarski 1930 or Tarski 1956), who proved that in the case of the first order classical logic, every countable set of formulae is semantically equivalent to an independent set. In fact, in 1923, he proved something more general, namely that any countable theory of $L_{\omega\omega}$ has an independent axiomatization. $L_{\omega\omega}$ denotes here the language of the first order classical logic which admits countable conjunctions and disjunctions, and infinite (but countable) sequences of quantifiers.

In 1965, the result was further generalized by Iegor Reznikoff to theories of any cardinality (Reznikoff 1965).

In general, proving independence of a set of axioms of a given theory is not an easy task.

Jan Łukasiewicz used many-valued matrices to establish independence of logical axioms in classical propositional logic (Łukasiewicz 1929). His system consists of three axioms:

1) $(p \to q) \to \big((q \to r) \to (p \to r)\big)$
2) $(\neg p \to p) \to p$
3) $p \to (\neg p \to q)$

together with the rules of modus ponens and of substitution.

The idea of proving the independence of the system consists in finding for every axiom some matrices hereditary with respect to the given primitive rules and such that all the axioms, except the chosen one, belong to their content.

For example, let us take a three valued matrix with the distinguished value 1 and propositional connectives ¬ and → defined in the following way:

p	$\neg p$
0	1
1	0
2	2

and

→	0	1	2
0	1	1	1
1	0	1	0
2	1	1	0

It is hereditary with respect to the rules of modus ponens and of substitution. Axioms 2) and 3) belong to its content, but taking a valuation associating 2 to p, 0 to q and 2 to r we obtain the value 0 for the formula constituting the first axiom.

Łukasiewicz, encouraged by his colleagues, Alfred Tarski, Adolf Lindenbaum, Jerzy Słupecki, Bolesław Sobociński and Mordechaj Wajsberg among them, investigated not only the full propositional calculus, with different sets of connectives as basic, but also many partial calculi, in particular the pure implicational calculus and the pure equivalential calculus in order to find not only their independent axiomatizations, but also the shortest and the simplest possible ones (Łukasiewicz 1961).

In particular, Łukasiewicz found a single axiom which, together with the rules of substitution and of detachment for equivalence, is sufficient for the equivalence logic. In 1925 Tarski showed that the pure implicational calculus can be based on a single axiom, the shortest possible form of which was discovered in 1936 by Łukasiewicz and Wajsberg, though the publication of the result had to wait until 1948 (Łukasiewicz 1948):

$$CCCpqrCCrpCsp$$

It is clear that any axiomatization consisting of a single axiom is independent.

To generalize Łukasiewicz's idea of proving independence, one can notice that in order to show the independence of a formula α from the set X of formulae it is enough to find such a property that belongs to all formulae from X but which is not true for α. In more advanced systems this idea is applied by constructing appropriate models. The good example is the independence of the axiom of choice or the continuum hypothesis from the Zermelo-Fraenkel set theory. The technique used there is called forcing (see e.g. Bell 2011).

Let us observe that in the both cases mentioned above, talking about independence we mean not only the impossibility of proving an axiom itself but also the impossibility of refuting it. In other words, we deal with a situation when a sentence is undecidable in a theory, or is absolutely independent from it. Intuitively, it means that an absolutely independent sentence represents entirely new information, which is not contained in the axioms.

Therefore, we can regard the absolute independence in classical logic as a natural generalization of the logical independence, and we say that a set S of formulae (of axioms, in particular) is absolutely independent if for any partition of S into sets S_1andS_2 the set $S_1 \cup \neg S_2$ is consistent, where $\neg S_2$ denotes the set of negations of all formulae from S_2.

It can be proved that in the first order classical logic every set of formulae for which the minimal cardinality of the set of axioms is a regular cardinal has an absolutely independent set of axioms. On the other hand, it is also possible to construct a countable set of formulae in the first order language with a finite number of primitive symbols without any absolute axiomatization (Grygiel 1990).

From the algebraic point of view, the existence of an absolute independent axiomatization of a classical theory T corresponds to the existence of an independent set of generators for the filter F corresponding to the theory T in the Lindenbaum algebra. Analogically, the existence of a logical independent axiomatization of the theory T corresponds to the existence of a filter independent set of generators for the corresponding filter F.

Both these algebraic notions are special cases of a general notion of algebraic independence introduced by Edward Marczewski in the late fifties (Marczewski 1958).

There are many equivalent ways of defining independence in algebra. We can say that the set X of elements of an algebra A is independent if the subalgebra generated by X is free in the variety generated by A. This means that independent sets are the basis of free algebras.

We can also formulate an equivalent definition operating on polynomials. The set X of elements of an algebra A is independent if and only if for any polynomials p and q of the algebra A and any system $b_1, \dots, b_n$of elements from X the equality $p(b_1, \dots, b_n) = q(b_1, \dots, b_n)$ implies the equation $p(a_1, \dots, a_n) = q(a_1, \dots, a)$ for every a $a_1, \dots, a_n$ from A.

Many problems concerning this notion of independence were intensively investigated by Marczewski and his followers: Mycielski, Narkiewicz, Świerczkowski, Głazek and others (so called Wroclaw Mathematical School), which resulted in more than 50 articles on the theory of free algebras only in the late fifties and sixties see e.g. Marczewski 1966). There are also some papers of Tarski concerning this topic (e.g. Jónson, Tarski 1961).

Marczewski's notion of independence contains, as specific cases, many notions of independence, considered in different areas of mathematics. Apart from previously mentioned filter independence corresponding to logical independence, the independence of vectors in linear spaces or the independence of subsets of boolean algebras.

A subset A of a boolean algebra B is independent if and only if for any system $a_1, \dots, a_n$ of different elements from A and any function $\epsilon: B \to B$ such that $\epsilon a \in \{a, -a\}$ for every $a \in A$ we have $\epsilon a_1 \wedge \dots \wedge \epsilon a_n \neq 0$. Independent subsets of boolean algebras have been intensively investigated not only in algebra (e.g. Sikorski 1960) but in topology as well (e.g. Koppelberg 1989).

The particular emphasis was given to a special question: given a boolean algebra B, for which cardinals κ does B have a free subalgebra of cardinality κ. Interesting answers to this question were provided by Shelah, Balcar and Franek (Shelah 1980, Balcar and Franek 1982).

As it was mentioned before, the question concerning the existence of an absolutely independent set of axioms of a given classical theory has its algebraical counterpart in boolean algebras, but is connected not with independent sets of generators of the whole algebra but with independent sets of generators of its filters.

The notion of independence in boolean algebras is also strictly connected with the notion of probabilistic independence.

It is well known that two events A and B are independent if the probability of the fact that they both occur simultaneously is equal to the product of probabilities of their occurrences. Two random variables, X and Y, are said to be independent if any event defined in terms of X is independent of any event defined in terms of Y. Algebraically, it means that they generate two independent σ-algebras. Two σ-algebras on a set X are independent if any element of one of them is independent from the set of elements of the other in the Boolean algebra $P(X)$.

To the independence of events, or rather elementary situations there were devoted some works of Wolniewicz, who dealt with them from the philosophical point of view. He tried to formalize, by applying algebraic tools, Wittgenstein's idea of logical atomism, assuming only two basic ontological properties of elementary situations - their atomicity and mutual independence (Wolniewicz 1999).

There is also an interesting link between quantum randomness and absolute independence in classical logic (which is a logical counterpart of algebraic independence in boolean algebras). It was shown (Paterek, Kofler and others 2010) that quantum states from a certain class encode mathematical axioms and that corresponding measurements test the truth-values of mathematical propositions. Quantum mechanics imposes an upper limit on the amount of information carried by a quantum state, limiting in that way the information content of the set of axioms. Whenever a proposition is absolutely independent of the axioms encoded in the state, the measurement associated to the proposition gives random outcomes. Otherwise, the measurement outcome is definite. This shines new light on the nature of quantum randomness, the roots of which are still not fully known.

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Logiczna niezależność, jej algebraiczne uogólnienie i zastosowania

Streszczenie

W pracy przedstawiamy krótką historię pojęcia logicznej niezależności, jej algebraicznego odpowiednika i jego uogólnienia do niezależności algebraicznej, a także podajemy przykłady wykorzystania niezależności w rozmaitych dziedzinach.

Słowa kluczowe: logiczna niezależność, niezależna aksjomatyka, niezależność algebraiczna, algebry wolne.