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# $k$-CONNECTIVITY

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**Abstract**
The notion of some connectedness is introduced and it is compared with usual connected sets.

Subsets of a topological space that are called $i$-connected have been introduced by J. Knop & M. Wróbel in [2] in 2006. The reason of introducing this notion is based upon the fact that a subset of the space $\mathbb{R}$ with nonempty interior is $i$-connected if and only if it is connected.

**Definition 7** (J. Knop & M. Wróbel – 2006) A subset $A$ of a topological space $X$ is called to be $i$-connected if it is connected and $\mathrm{Int}\,(A)$ is connected.

We want to define the so called $k$-connected subsets of a topological space, which are a little different from $i$-connected sets, but have some interesting properties.

**Definition 8** *A subset $A$ of a topological space $X$ is called to be $k$-connected if its interior is connected and $A \subset \overline{\mathrm{Int}\,(A)}$.*

Of course, each $k$-connected set is $i$-connected and connected as well, but not conversely.

Moreover, let us remark that each open connected set is $k$-connected as well.

It is quite easy to see that for Euclidean space of real numbers $\mathbb{R}$, a subset $A$ with a nonempty interior is $k$-connected if and only if it is connected.

As we could observe, $k$-connected sets are similar to $i$-connected ones, so it is worth to compare these kinds of sets.

**Theorem 11** *A subset $A$ of a topological space is $k$-connected if and only if it can be represented in the form*

$$A = B \cup C$$

*such that $C$ is open and connected and*

$$B \cap C = \emptyset, \quad B \subset \overline{C}.$$

P r o o f. 1. Suppose first that a set $A$ is $k$-connected. Let $C = \mathrm{Int}\,(A)$ and $B = A \setminus C$. Then, of course, $\mathrm{Int}\,(B) = \emptyset$ and $B \cap C = \emptyset$. Since $A \subset \overline{\mathrm{Int}\,(A)}$, then $B \subset \overline{C}$.

2. Suppose now that a set $A$ has the form $A = B \cup C$, where $C$ is open and connected and

$$B \cap C = \emptyset, \quad B \subset \overline{C}.$$

Then

$$C = \mathrm{Int}\,(C) \subset \mathrm{Int}\,(A)\,,$$

consequently

$$\overline{C} \subset \overline{\mathrm{Int}\,(A)}$$

and

$$A = B \cup C \subset \overline{C} \cup C = \overline{C} \subset \overline{\mathrm{Int}\,(A)}.$$

Two considered possibilities complete the proof. □

**Theorem 12** *If $A$ is a $k$-connected subset of a topological space and $C$ fulfils the following inclusions*

$$A \subset C \subset \overline{A},$$

*then $C$ is also a $k$-connected set.*

P R O O F. Since $\mathrm{Int}\,(A)$ is a connected set and

$$\mathrm{Int}\,(A) \subset \mathrm{Int}\,(C) \subset \mathrm{Int}\,\left(\overline{A}\right) \subset \overline{A} \subset \overline{\overline{\mathrm{Int}\,(A)}} = \overline{\mathrm{Int}\,(A)},$$

then $\mathrm{Int}\,(C)$ is a connected set. Moreover,

$$C \subset \overline{A} = \overline{\overline{\mathrm{Int}\,(A)}} = \overline{\mathrm{Int}\,(A)} \subset \overline{\mathrm{Int}\,(C)},$$

which proves the theorem. □

**Theorem 13** *If $\{A_s : s \in \mathcal{S}\}$ is a class of $k$-connected subsets of a topological space and there exists an index $s_0$ in $\mathcal{S}$ such that*

$$\mathrm{Int}\,(A_s) \cap \mathrm{Int}\,(A_{s_0}) \neq \emptyset$$

*for all $s$ from $\mathcal{S}$, then the set $\bigcup\limits_{s\in\mathcal{S}} A_s$ is also a $k$-connected subset of $X$.*

P R O O F. Each of the sets $\mathrm{Int}\,(A_s)$ is connected, so is $A_s$. In view of our assumptions, the sets $\bigcup\limits_{s\in\mathcal{S}} A_s$ and $\bigcup\limits_{s\in\mathcal{S}} \mathrm{Int}\,(A_s)$ are connected as well. Since each of the sets $A_s$ is $k$-connected, then $A_s \subset \overline{\mathrm{Int}\,(A_s)}$ and applying standard properties of interior and closure operations one can infer from the definition of $k$-connected sets that

$$\mathrm{Int}\left(\left(\bigcup_{s\in\mathcal{S}} \mathrm{Int}\,(A_s)\right)\right) = \bigcup_{s\in\mathcal{S}} \mathrm{Int}\,(A_s) \subset \bigcup_{s\in\mathcal{S}} \overline{\mathrm{Int}\,(A_s)} \subset \overline{\bigcup_{s\in\mathcal{S}} \mathrm{Int}\,(A_s)}.$$

From this and connectedness of the set $\bigcup\limits_{s\in\mathcal{S}} \mathrm{Int}\,(A_s)$ we can infer that the set $\mathrm{Int}\left(\left(\bigcup\limits_{s\in\mathcal{S}} A_s\right)\right)$ is connected.

Moreover,

$$\bigcup_{s\in\mathcal{S}} A_s \subset \bigcup_{s\in\mathcal{S}} \overline{\mathrm{Int}\,(A_s)} \subset \overline{\bigcup_{s\in\mathcal{S}} \mathrm{Int}\,(A_s)} \subset \overline{\mathrm{Int}\left(\left(\bigcup_{s\in\mathcal{S}} A_s\right)\right)},$$

which completes the proof. □

As usual, one can ask whether a continuous image of a $k$-connected set is also $k$-connected. The following theorem gives the positive answer under some additional condition.

**Theorem 14** *If $X$ and $Y$ are topological spaces and a map $f : X \longrightarrow Y$ is an open and continuous injection of $X$ into $Y$, then $f(A)$ is a $k$-connected subset of $Y$ whenever $A$ is a $k$-connected subset of $X$.*

P R O O F. If $A$ is a $k$-connected subset of the space $X$, then it can be represented in the form

$$A = B \cup C,$$

where $C$ is open, connected and

$$B \cap C = \emptyset, \quad B \subset \overline{C}.$$

Then

$$f(A) = f(B \cup C) = f(B) \cup f(C).$$

The set $f(C)$ is open, connected and

$$f(B) \cap f(C) = \emptyset, \quad f(B) \subset f\left(\overline{C}\right) \subset \overline{f(C)}.$$

In view of Theorem 11, the set $f(A)$ is $k$-connected. □

There is no difficulty to construct topological spaces and a continuous surjection for which the image of $k$-connected sets is not $k$-connected. So, openness of such a map cannot be omitted in the previous theorem.

If a topological space $X$ fulfils the condition: for each element $x$ of $X$ there exists an open and connected set $U$ such that $x \in U$, then we can define the notion of $k$-component of a point. Theorem 13 allows us to consider the biggest $k$-connected set containing a point $x$. It is called the *$k$-component* of this point in topological space $X$. It happens that in such spaces $k$-components coincide with components (in the usual sense). Let us notice that this condition is a bit weaker than local connectedness of the space.

**Theorem 15** *If $X$ is a topological space such that for each element $x$ of $X$ there exists an open and connected set $U$ such that $x \in U$, then $k$-component of any point $x$ is equal to the component of this point (in the usual sense).*

P R O O F. Since each $k$-connected set is connected (in the usual sense), then $k$-component of a point in any topological space is contained in the (usual) component of that point. It is left then that in topological spaces fulfilling our condition, the component of a point is contained in its $k$-component. Let $C$ be the component of a point $x_0$. For each point $x$ from the set $C$ there exists an open and connected set $U_x$ such that $x \in U_x$. From the definition of components it follows that $U_x \subset C$ for each point $x$ from $C$. Thus the component $C$ is a union of all sets $U_x$, $x \in C$. Then the set $C$ is open, hence it is $k$-connected and containing the component of the point $x_0$. □

Now we will consider some sufficient conditions for topological spaces in which the class of $k$-connected sets and the class of connected sets with nonempty interior coincide. Let us denote this class of topological spaces by $\mathscr{K}$.

**Theorem 16** *If $X$ is a topological space from $\mathscr{K}$, then*

$$x \in \operatorname{Int}\left(\left(\overline{U} \cup \overline{V}\right)\right)$$

*for each disjoint open and connected sets $U$ and $V$ and $x$ belonging to $\overline{U} \cap \overline{V}$.*

P R O O F. Let us suppose in the contrary that there exist disjoint, open and connected sets $U$, $V$ and a point $x$ such that

$$x \in \overline{U} \cap \overline{V}, \quad x \notin \operatorname{Int}\left(\left(\overline{U} \cup \overline{V}\right)\right).$$

Since the sets $U$, $V$ are not separated from $\{x\}$, then the set $E$, where $E = U \cup V \cup \{x\}$, is connected. Its interior equals $U \cup V$, which is not connected. In this way we obtained a connected set which is not $k$-connected. Contradiction. □

Similarly, one can prove the next theorem:

**Theorem 17** *If $X$ is a topological space from $\mathscr{K}$ and $U$, $V$, and $W$ are pairwise disjoint open and connected subsets of $X$, then*

$$\overline{U} \cap \overline{V} \cap \overline{W} = \emptyset.$$

Let us remind the notions of cut points and strong cut points of a topological space.

**Definition 9** *Let $X$ be a connected topological space. A point $x$ from $X$ is called a cut point if the set $X \setminus \{x\}$ is not connected.*

**Definition 10** *Let $X$ be a connected topological space. A point $x$ from $X$ is called a strong cut point if the set $X \setminus \{x\}$ has two components.*

As a corollary we can obtain the following theorem:

**Theorem 18** *If $X$ is a topological space from $\mathscr{K}$ and $x$ is a cut point of the space $X$, then $X \setminus \{x\}$ has two components.*

We can formulate this theorem in other words: Each cut point of a space $X$ from the class $\mathscr{K}$ is a strong cut point.

# References

[1] J.L. Kelley. *General Topology.* Springer, New York - Heidelberg - Berlin 1955.

[2] J. Knop, M. Wróbel. Some properties of $i$-connected sets. *Annales Academiae Paedagogicae Cracoviensis, Studia Mathematica,* VI, Folia 45, 51–56, 2007.