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# WHY LOGARITHMS?

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**Abstract.** In 16th and 17th century, the need for speed in complex calculation spurred the invention of a powerful mathematical tool known as LOGARITHM. The reduction of multiplication and division to addition and subtraction (likewise the reduction of a complex mathematical structure to more simple ones) is in the spirit of "prosthaphaeretic rulesóf ancient Greeks. We discuss some mathematical ideas related to logarithms and present some historical notes.

Rarely in the history of science has an abstract mathematical idea been received more enthusiastically by the entire scientific community than the invention of logarithms. The sixteenth and early seventeenth centuries saw an enormous expansion of scientific knowledge in every field. Discoveries in geography, physics and astronomy, rapidly changed man's perception of the universe: Copernicus's heliocentric system, Magellan's circumnavigation of the globe in 1521, the new world map published in 1569 by Gerhard Mercator, inventions and needed new knowledge in numerical computation, in formulated new physics laws, for example mechanics (Galileo Galilei) and astronomy (Johannes Kepler, his three laws of planetary motion).

These developments involved an ever increasing amount of numerical data, forcing scientists to spend much of their time doing numerical computation. The times called for an invention that would free scientists once and for all from this burden. Napier took up the challenge. We have no account of how Napier first stumbled upon the idea that would ultimately result in his invention. He was well versed in trigonometry, e.g. he was familiar with

$$\sin A \sin B = \frac{1}{2}\left[\cos\left(A-B\right) - \cos\left(A+B\right)\right]$$

This formula and similar ones for $\cos A \cos B$ and $\sin A \sin B$ are known as the *prosthaphaeretic rules*, from the Greek word meaning ąddition and subtraction". Their importance lays in the fact that the product of two trigonometric expressions such as $\sin A \sin B$ could be computed by finding the sum or difference of other trigonometric expressions, in this case $\cos(A - B)$ and $\cos(A + B)$. Since it is easier add and subtract than to multiply and divide, these formulas provide primitive system of reduction from one arithmetic operation to another, simpler. Roughly, this was originally the idea of John Napier.

## 1. Invention of logarithms

This idea is better illustrated for the terms of a geometric progression, i.e. a sequence of numbers with a fixed ratio between successive terms. For example, the sequence $1, 2, 4, 8, 16, \ldots$ is a geometric progression with the common ratio 2. If we denote the common ratio by $q$, starting with 1, the terms of progression are $1, q, q^2, q^3, \ldots$ (note that the $n$-th term is $q^{n-1}$).

Long before Napier's time, it had been noticed that there exists a simple relation between the terms of a geometric progression and the corresponding exponents. Nicola Oresme in his book *De proportionibus proportionum* generalized some rules for combining proportions in year 1360. Nowadays expressed as $q^m q^n = q^{m+n}$ and $(q^m)^n = q^{mn}$. These relations are exactly formulated by Michael Stifel in his book *Arithmetica integra* as follow : "*If we multiply any two terms of the progression* $1, q, q^2, \ldots,$ *the result would be the same as if we had added the corresponding exponents... dividing one term by another term is equivalent to subtracting their exponent.*"

The problem arises, if the exponent of the denominator is greater than that of the numerator. To get around this difficulty, we simply define $q^{-n} = \frac{1}{q^n}$ and $q^0 = 1$, so that $\frac{q^3}{q^5} = q^{-2} = \frac{1}{q^2}$ and $\frac{q^3}{q^3} = q^0 = 1$. With this definition, we can extend a geometric progression indefinitely in both directions, $\ldots, q^{-3}, q^{-2}, q^{-1}, q^0, q^1, q^2, \ldots$. Each term is a power of the common ratio $q$, and that the exponents $\ldots, -3, -2, -1, 0, 1, 2, \ldots$ form an arithmetic progression. This relation is the key idea behind logarithms. Stiffel had in mind only integer values of the exponent, but Napier's idea in his book *Mirifici Logarithmorum Canonis descriptio* was to extend it to a continuous range of value's .

His thoughts proceeded as follows: If we could write any positive number as a power of some given fixed number then multiplication and division of positive numbers would be equivalent to addition and subtraction of their exponents. We illustrate the idea with number 2 as the base.

Suppose we need to multiply 16 by 64. We look in the table for the exponents corresponding to 16 and 64 and find them as 4 and 6. Adding these exponents gives us 10. We now reverse the process, looking for the number

| $n$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $2^n$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |

whose corresponding exponent is 10. This number is 1024 and we have the desired answer. As the next example, suppose we want to find $4^4$. We find the exponent corresponding to 4, namely 2 and this time multiply it by 4 to get 8, then look for the number whose exponent is 8 and find it to be 256. And indeed $4^4 = \left(2^2\right)^4 = 2^8 = 256$.

Of course, such an elaborate scheme is unnecessary for computing strictly with integers. But, for this to happen, we must first fill in the large gaps between the entries of our table. We can do this in one of two ways:

- using fractional exponents
- choosing for the base a number small enough so that its powers will grow reasonably slowly.

## 2. Napier logarithms

Napier choose the second option. A question arose, how to choose the base so that a change of exponent causes a small change of powers of the base. It seems that it should be a number close to 1. Napier decided $0,9999999$ or $1-10^{-7}$. Napier spent twenty years of his life to complete the task, that it will do the job. His initial table contained just 101 entries, starting with $10^7$ and followed $10^7\left(1-10^{-7}\right) = 9999999$, then $10^7\left(1-10^{-7}\right)^2 \doteq 9999998$ and so on up to $10^7\left(1-10^{-7}\right)^{100} \doteq 9999900$. The difference between two sides is only $0,000495$, which we neglect. Each term being obtained by subtracting from the preceding term its $0,9999999 \doteq 10^7\left(1-10^{-7}\right)^{i+1} - 10^7\left(1-10^{-7}\right)^i = \left(1-10^{-7}\right)^i$. He then repeated the process again, starting once with $10^7$, but this time taking as his proportion the ratio of the last number to the first in the original table, that is $\frac{9999900}{10000000} = 0,99999$ or $1-10^{-5}$. This second table contained 51 entries. The first was $1-10^{-7}$, followed $10^7\left(1-10^{-5}\right) = 9999900$, the last being $10^7\left(1-10^{-5}\right)^{50} \doteq 9995001$. A third table with 21 entries using the ratio $\frac{9995001}{10000000}$, the last entry in this table was $10^7\left(0,9995\right)^{20} = 9900473$. Finally, from each entry in this table Napier created 68 additional entries using the ratio $\frac{9900473}{10000000} \approx 0,99$ and the last entry then was $10^7 0,99^{68} \approx 4998609$, it is roughly half the original number.

In modern notation, this amounts to says that if $N = 10^7\left(1-10^{-7}\right)^L$, then the exponent $L$ is the Napier logarithm of $N$. Napier's definition of logarithms was different in several respects from the modern definition (introduced in 1728 by Leonhard Euler). If $N = b^L$, where $b$ is a fixed positive number other than 1, then $L$ is the logarithm (with the base $b$) of $N$. Hence

$L = 0$, Nap log $10^7 = 0$

$L = 1$, Nap log $9999999 = 1$, etc.

Notice that the computations using Napier logarithm are more complicated then those using the modern logarithms. for example :

If $L_1 =$ Nap log $N_1$, hence $10^7 N_1 = 10^7 \left(1 - 10^{-7}\right)^{L_1}$ and

$L_2 =$ Nap log $N_2$, hence $10^7 N_2 = 10^7 \left(1 - 10^{-7}\right)^{L_2}$,

then $\frac{N_1 N_2}{10^7} = 10^7 \left(1 - 10^{-7}\right)^{L_1 + L_2}$, $nL_1 =$ Nap log $\frac{N_1^n}{10^{7(n-1)}}$.

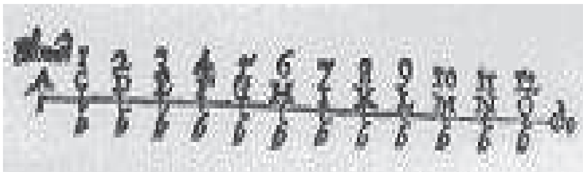
Relationship between the modern logarithm and the Napier logarithm :

Nap log $N = \frac{\ln \frac{N}{10^7}}{\ln\left(1 - 10^{-7}\right)}$,

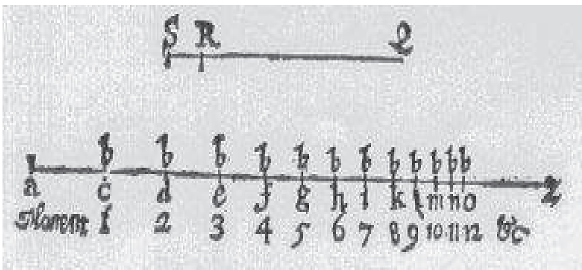
Nap log $N_1 N_2 =$ Nap log $N_1 +$ Nap log $N_2 + \frac{\ln 10^7}{\ln\left(1 - 10^{-7}\right)}$.

## 3. Geometric definition of the Napier logarithm

The principles of his work explained in geometric terms have been presented in first in the article about logarithms: *Mirifici Logarithmorum Canonis descriptio* Assume that a point P moves along ACZ (starting from A) with the



uniform speed B. Now, in the first moment, let P move from A to C, in the second moment from C to D, etc. Let SQ be a line segment and let AZ be a halfline, see picture.



Let a point R start from S and move along SQ with variable speed decreasing in proportion to its distance from Q. During the same time let a point P start from A and move along ACZ with uniform speed. Napier called this variable distance AP the logarithm of the distance RQ.

*"The Logarithme therfore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed and the beginning equally swift."*

Napier's geometric definition is, in agreement with the numerical description given above. Let |RQ|$=x$ and |AP|$=y$. If |SQ| is taken $10^7$ and if the initial speed R is also taken $10^7$, then in modern calculus notations we have $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = 10^7$. The initial boundary condition are $x_0 = 10^7$ and $y_0 = 0$. Then $\frac{dy}{dx} = -\frac{10^7}{x}$ or $y = -10^7 \ln cx$, where $c$ is found from the initial condition $c = 10^{-7}$. Hence $y = -10^7 \ln \frac{x}{10^7} = 10^7 \log_{\frac{1}{e}} \frac{x}{10^7}$

## 4. Conclusion

The creation of the idea of logarithm by Napier (connection greek's words $\lambda o\gamma o\varsigma$ - ratio, $\alpha\rho\iota\tau\mu o\zeta$ - number) is a revolutionary milestone in history. His invention was quickly adopted by scientists all across Europe and even in faraway China. Henry Briggs (professor of geometry at Gresham College in London) was impressed by the new invention and has said to Napier : *"My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz the logarithm ..."*

At that meeting, Briggs proposed two modifications that would make Napier's table convenient

to have the logarithm of 1, rather than of $10^7$, equal to 0 and

to have the logarithm of 10 equal an appropriate power of 10.

The beauty and the power of logarithm can be presented for example by computing $x = \sqrt[3]{\frac{493{,}823{,}67^2}{5{,}104}}$. To perform the computation it suffices to use "the table of logarithms from 10 to 52".

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