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# TEACHER'S STUDIES STUDENTS DIFFICULTIES CONCERNING THE GENERALIZATION OF THE CONCEPT OF THE RIEMANN INTEGRAL

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**Abstract.** One of the most fundamental concepts of the mathematical analysis is the Riemann integral. For a teacher of mathematics the concept of the integral is important because of the connections with the Jordan measure which is considered in the elementary geometry. Besides the Riemann integral the course of mathematical analysis includes multiple integrals, line integrals and surface integrals. In this paper we present the results of our research concerning the difficulties of students in noticing mutual connections between different kinds of integrals.

## 1. Introduction

One of the most fundamental concepts of mathematical analysis is the Riemann integral. It has found many important applications in both mathematics and related sciences, for example physics. In the syllabus of mathematical analysis that has been effective in the Pedagogical University of Cracow during recent years the integral calculus comes in the first and third years of studies, with the reservation that for the first year it involves integration of functions of one variable and for the third year - integration of functions of several variables. The third-year curriculum additionally includes multiple, oriented and non-oriented line and surface integrals as well as the concept of an integral with respect to a measure, and in particular the concept of the Lebesgue measure and integral. The order of familiarizing students with these concepts can be different. Some lecturers follow the differential calculus of functions of several variables with the theory of the Lebesgue measure and integral and only then do they proceed to multiple, line and surface integrals. With this approach it is possible from the general theory of an integral with respect to a measure to specify properties of this concept in a specific model, which is the space $\mathbb{R}^n$ with the Lebesgue measure therein specified.

The modified teaching *standards for fields of study*, which were published by the Central Council of Higher Education on its website on 19$^{\text{th}}$ February 2007 (www.rgsw.edu.pl/files/active/0/matematyka20070210), provide with reference to Mathematics that the curriculum of first degree studies should cover the differential and integral calculus of functions of one variable and several variables. According to the publication such generalizations as an integral with respect to a measure are moved to the curriculum of second degree studies.

The concept of the Riemann integral, referred to at the beginning of the article, is an important item in the curriculum of a teacher's field of study, since what matters for a teacher of mathematics is its connection with the concept of the Jordan measure, which is covered in school at a predefinition level. Please note that, if an integrand is nonnegative in the interval $P$, then its integral on this interval is equal to the area of a *curvilinear trapezium* formed by the graph of the function $f$, the $x$-axis and its perpendicular lines which intersect the interval boundaries, and so it is the Jordan measure of a certain area.

While defining the above-mentioned integrals it is the so-called *integration process* that comes to the foreground. It consists in considering a normal sequence of partitions of a set (a segment, a curve arc, a regular area) in $\mathbb{R}^n$ that an integrand is specified on, which is subsequently followed by constructing sequences of lower sums and sequences of upper sums, or alternatively a sequence of approximate sums*. The common limit of sequences of lower and upper sums, as long as it does not depend on a normal sequence of partitions, is called a (definite, line, surface) integral of a given function on a given set. Such integral can also be defined as a limit of a sequence of approximate sums, as long as it does not depend on a normal sequence of partitions and the way of selecting intermediate points from particular domain subsets.

In order to define an oriented line and surface integral, orientation on a curve or a surface needs to be specified first and followed by applying the integration process for the properly determined integrands. For this reason the concepts in question rank among those analogous to the concepts of the relevant non-oriented integrals and, therefore, are not the subject matter of this paper.

*A lower (upper) sum is defined as a sum of products of function minima (maxima) for each subset that the function domain has been divided into and measures of relevant subsets. An approximate sum is defined as a sum of products of function values in any point of each subset that the function domain has been divided into and measures of relevant subsets.

Long-standing observations of students during their classes on mathematical analysis point to the fact that the above-referenced subject matter is not popular with them. The reasons for it can be the following:

- too high complexity of calculations needed to perform the integration process;
- areas of integrals' application that are remote from the interests of students of mathematics;
- not properly shaped spatial imagination of students;
- failure to perceive analogies between different types of integrals and lacking skills of generalizing these concepts.

In order to find an answer the question relating to students' difficulties in generalizing the concept of a definite integral, a questionnaire was developed and conducted among third-year students of mathematics. The answers to the questionnaire questions as provided by research subjects were subsequently analyzed. Additionally, the attitude and behaviours of the students included in the research were observed during their classes on mathematical analysis.

## 2. On certain generalizations of the Riemann integral

The ability to generalize and perceive generalizations is an important component of mathematical activity, essential to study mathematics (Krygowska, 1986). It is discussed in didactics of mathematics in the context of concepts and theorems. While developing new concepts or theorems there are two ways that can be followed, which H. Siwek defines as follows:

- *from detailed examples to general concepts and thus to formulating theorems and proving them on a high level of generality,*
- *from general concepts and theorems of a given theory to examples and counterexamples which reflect definitions and to detailed theorem cases* (Siwek, 2005, p. 290).

These could be briefly described as *bottom up* and *top down* approaches. This paper regards the first type and, due to its size, focuses on the activity of generalizing concepts. Generalizing theorems will be the subject matter of the next article. In the context of developing concepts the *bottom up* approach can be organized in two ways. A. Z. Krygowska describes them in the following way: *Generalization of mathematical concepts by a student himself ( ... ) can* *be developed in such a way that the student either discovers a superiority relationship between two concepts he is already familiar with or consciously and deliberately constructs a concept superior to the one he is already familiar with* (Krygowska, 1977, p. 93). In didactics the former is known as generalization *through recognition*, while the latter is called generalization *through construction*. Both types have been presented during the lectures on mathematical analysis.

Multiple, non-oriented line and surface integrals are determined similarly to the Riemann integral of a function of one variable on an interval $[a, b]$. In fact the diversity of these integrals originates from increasing the dimension of the space where the integrand domain occurs. Please note that a function of two variables has a subset of the set $\mathbb{R}^2$ as its domain. It this is a rectangle, i.e. the Cartesian product of two intervals contained in the $\mathbb{R}$ or a plane curve understood as a homeomorphic interval image, then, by applying the *integration process*, referenced in paragraph 1, a double integral on the rectangle or a non-oriented line integral is obtained. This is an example of generalization *by recognition* as students learn all these concepts independently from one another and then their attention is called to the superiority relationship between these concepts, which is performed by analyzing the space dimension and the manner of constructing concepts.

If a function $f$ has a regular set $A$ as its domain, i.e. a set whose boundary consists of a finite number of curves of $y = y(x)$ or $x = x(y)$, then, in order to define a double integral of the function $f$ on the set $A$, the following construction is performed:

- the set $A$ is inscribed into the rectangle $P$ whose sides are parallel to the axes of the coordinate system,

- a new function $g$ is defined which is equal to the function $f$ on the set $A$ and takes zero on the set $P \setminus A$.

In this case a double integral over the set $A$ of the function $f$ is called a double integral of the function g over the rectangle $P$. This can be taken an example of generalization *through construction* as the approach which leads to a new concept (here to the concept of a double integral of the function $f$ over the regular set $A$) is based on the previously known concept, i.e. the concept of a double integral over a rectangle. The above-described construction is performed on purpose so as to refer to a situation which is already known to the students.

The theory of an integral in the space $\mathbb{R}^3$ considers functions which can have plane areas, curves or regular surface sheets as their domains. By applying the above-mentioned *integration process* independently for each of these functions one can reach the concept of a triple non-oriented line or surface integral. In this case again we can talk about generalization *through recognition*.

These different ways of generalization were brought to students' attention during the lectures and classes. The aim of these didactic techniques was to draw students' attention to the connections between relevant concepts, to point out which concept is a generalization of which one. For this reason they were meant to facilitate overcoming difficulties related to the concepts of particular integrals.

## 3. Research description and result analysis

The research was conducted in the Pedagogical University of Cracow in the academic year 2005/2006. It included 30 third-year students of mathematics.

In order to work out how students perceive different types of integrals and connections between them they were asked to express their own opinion on definitions and properties of integrals, to indicate these fragments of their definitions that they find difficult or to make a list of applications of relevant integrals (including the content of the tasks which included such integrals).

While analyzing the research results it became evident that the students found the non-oriented line integral and multiple integrals as the easiest, whereas the oriented and non-oriented surface integrals as the most difficult. The reason for might be the fact that the non-oriented line integral is a generalization of the Riemann integral, since, if a plane curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is contained in either axis of the coordinate system, then a non-oriented line integral over such curve is reduced to a definite integral of a function of one variable. What arises from this is that the students might have noticed this generalization. Another reason why the students recognized the non-oriented line integral as the easiest might be not too high complexity of required calculations, which actually boil down to calculating only one single integral.

The students deemed multiple integrals as more difficult than the non-oriented line integral. Some of them wrote that the fundamental difficulty in applying these integrals for different geometrical problems is the lack of a properly shaped spatial imagination. This is proved correct by observations made during the classes, which show that a great number of students could not imagine surfaces denoted by such equations as $z = x^2 + y^2$ or $z = x^2 - y^2$. Their difficulty in dealing with such tasks might originate from algebraization of geometric problems. There is no doubt, however, that problems of spatial imagination are a significant barrier that needs to be overcome while solving tasks.

The students found surface integrals as the most difficult concept. As pointed out in paragraph 2, the non-oriented surface integral is a generalization of the multiple integral specified in a plane area in the case of a surface sheet in $\mathbb{R}^3$, but this connection was not always perceived.

The questionnaire also asked the students to specify which parts of the definitions of the above-mentioned integrals caused them special difficulty. The analysis of their responses points to a few reasons:

a) Not understanding a normal sequence of partitions of a relevant set as well as the manner of selecting intermediary points and constructing a sequence of approximate sums;

b) Not understanding concepts crucial to the definition of an integral; concepts of a normal area and a regular surface sheet were mentioned here.

Please note that the first type of difficulties was pinpointed also by the students themselves when the understanding of the concept of a single integral was being researched (Powązka, 2007). The second type of difficulties probably results from an inability to visualize patterns describing boundaries of areas on which integration is performed. The research showed that 20 respondents failed to notice connections between relevant types of integrals. With regard to the non-oriented line integral only three research subjects noticed that it is a generalization of a single integral and with regard to the surface integral only two persons indicated that it is a generalization of a double integral. Worth mentioning is also the fact that one of the research subjects found a non-oriented line integral to be a generalization of an oriented line integral. Two other students formed a similar conclusion with regard to surface integrals.

Although this article deals with problems related to concepts and issues concerning application of theorems will be discussed in the next article, it is worthwhile hinting that difficulties in calculating integrals (and therefore application of relevant theorems) have a negative effect on understanding them. For example, as the analysis of the research data reveals, the difficulty in calculating multiple integrals may lie in the application of the Fubini's theorem and the necessity to calculate at least two single integrals. This task is more effort-consuming than calculations required for a non-oriented line integral. Sometimes it also becomes necessary to apply the change of variable theorem in a multiple integral, which admittedly simplifies calculations but requires a proper transformation of the integration area. However, finding the correct transformation can be difficult for a student whose spatial imagination is not properly shaped. Yet, it does not seem true to state that the complication of calculations significantly hinders the perception of connections between concepts.

## 4. Final notes

The research and observation of the students allow constructing some hypotheses concerning the students' perception of the concept of a single integral in the context of generalizing. At the same time, the hypotheses express difficulties which the research students had while applying the indicated concept. The research results seem to point to the following:

[1] The students identified the concept of a non-oriented line integral with the Riemann integral; they did so most probably because of the similarity of symbols used for both integrals, but they did not always use these denotations correctly.

[2] The research subjects associated non-oriented surface integrals with the symbol of a double integral (this was observed in the part of the research on theorems).

[3] The respondents showed poor spatial imagination, which caused significant difficulty in describing the integration area.

[4] It seems that generalization through recognition was easier for the students than generation through construction.

In the context of the last hypothesis it is worth to quote Z. Krygowska: "It is not difficult ( ... ) to notice that generalization *through recognition* is something psychologically different from generalization *through construction*"(Krygowska, 1977, p. 94). It would be worthwhile performing further research in order to determine whether the last hypothesis is really correct in the context of the concept of an integral. Moreover, interesting would be researching whether generalization *through recognition* is a more complicated process for students than generalization *though construction* also in the context of other mathematical concepts.

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