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GEOMETRIC CONSTRUCTIONS AS PROBABILISTIC SPACES CONSTRUCTIONS

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Abstract. The study proposes a visualization of the discrete probabilistic space idea as well as its construction.

1. The discrete probabilistic space and its production

Definition 1-1 [THE PROBABILITY DISTRIBUTION OVER A SET-CLASS] Let $s \in \mathbb{N}_2$ and $\Omega = \{\omega_1, \omega_2, \omega_3, \dots, \omega_s\}$. *The distribution of probability over the* Ω *class* is any function $p : \Omega \longrightarrow \mathbb{R}$, which is non-negative and such, that:

$$p(\omega_1 + p(\omega_2 + p(\omega_3) + \cdots + p(\omega_s) = 1.$$

If $p(\omega_1 = p(\omega_2) = p(\omega_3) = \cdots = p(\omega_s) = \frac{1}{s}$, we call the function p *the classical distribution of probability over the* Ω *class.*

Definition 1-2 [THE DISCRETE PROBABILISTIC SPACE] Let Ω be any s-element class, and p be the distribution of probability over that class. Let $\mathcal{Z} = 2^{\Omega}$. We shall consider a function $P : \mathcal{Z} \longrightarrow \mathbb{R}$, where

$$P(A) = \begin{cases} 0, & \text{when } A = \emptyset, \\ p(\omega), & \text{when } A = \{\omega\}, \\ \displaystyle\sum_{\omega \in A} p(\omega), & \text{when } A \text{ is a class of at least two elements.} \end{cases}$$

It is not very difficult to prove, that a trio $(\Omega, \mathcal{Z}, P)$, which evolved from a pair (Ω, p), is a probabilistic space in terms of its axiomatic definition (see [13], p. 124). Such a trio, or – which is the same – pair (Ω, p) we call *the discrete probabilistic space.*

Definition 1-3 [THE CARTHESIAN PRODUCT OF DISCRETE PROBABILISTIC SPACES] Let us assume, that (Ω_1, p_1) and (Ω_2, p_2) are discrete probabilistic spaces, that $\Omega_{1-2} = \Omega_1 \times \Omega_2 = \{(x, y) : x \in \Omega_1 \wedge y \in \Omega_2\}$ and

$$p_{1-2}(x, y) = p_1(x) \cdot p_2(y) \text{ for every } x \in \Omega_1 \text{ and } y \in \Omega_2.$$

We shall call a pair (Ω_{1-2}, p_{1-2}) *the Cartesian product of the* (Ω_1, p_1) *and* (Ω_2, p_2) *discrete probabilistic spaces* and we shall note it as $(\Omega_1, p_1) \times (\Omega_2, p_2)$. The product $(\Omega, p) \times (\Omega, p)$ shall be called *carthesian square* and noted as $(\Omega, p)^2$.

It is easy to prove, that the Cartesian product of discrete probabilistic spaces is a discrete probabilistic space as well.

2. Figures with grids as geometric presentations of discrete probabilistic spaces

Definition 2-1 [A GRID IMPOSED ON A FIGURE WITH A POSITIVE AREA] Let F mean a set of those figures on a plane, which have an area. If $A \in F$, $m_2(A)$ means the area of the A figure, and $int(A)$ means its inside.

Let us assume, that $S = \{1, 2, 3, \ldots, s\}$, where $s \in \mathbb{N}_2$, and that

(i) $F \in \mathcal{F}$ and $0 < m_L(F) < +\infty$,

(ii) $\forall j \in S : [F_j \subset F \wedge F_j \in \mathcal{F} \wedge m_L(F_j) > 0]$,

(iii) $\forall j, k \in S : [j \neq k \Longrightarrow int(F_j) \cap int(F_k) = \emptyset]$,

(iv) $F_1 \cup F_2 \cup \ldots \cup F_s = F$.

The class $\Omega = \{F_j : j \in S\}$ we call *a grid imposed on the figure* F, and each element of that class (each figure F_j) we call the mesh.

The Tangram (see [16] and Fig. 1) was created by imposing on a square a grid of triangular meshes T_1, T_2, T_3, T_4, T_5 a square mesh K and a parallelogram R. Those meshes are called *tans*. If the base square's area is 1, the function p_T, which attributes an area to each tan, is a distribution of probability over the class $\Omega_T = \{T_1, T_2, T_3, T_4, T_5, K, R\}$. So a pair (Ω_T, p_T) is a non-singular discrete probabilistic space. Defining the p_T function is a geometrical task.

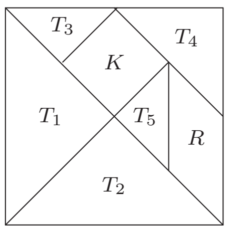


Fig. 1 The Tangram of seven tans

To define a discrete probabilistic space we need to impose on a square that has an area of 1 a grid consisting of figures that have an area.

■ [A PROBABILISTIC SPACE GENERATED BY A GRID IMPOSED ON A FIGURE WITH A POSITIVE AREA] Let a class $\{F_j : j \in S\}$ be a grid imposed on a figure F with a positive area, where $S = \{1, 2, 3, \ldots, s\}$ and $s \in \mathbb{N}_2$. Function $p : \{F_j : j \in S\} \longrightarrow \mathbb{R}_0$, where

$$p(F_j) = \frac{m_2(F_j)}{m_2(F)}, \tag{1.0..12}$$

is a distribution of probability over the class $\Omega = \{F_j : j \in S\}$. A pair (Ω, p) is called *a discrete probabilistic space generated by a grid imposed on a figure F*.

Figure 2 shows three probabilistic spaces generated by grids imposed on three figures having a positive area. To define each of those spaces as a pair (a class and a distribution of probability over that class) means to do a task of counting the areas of certain figures.

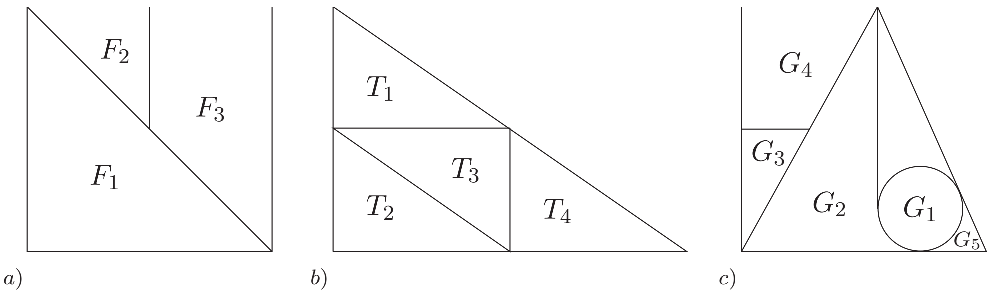


Fig. 2 Figures with finite grids as a presentation of finite probabilistic spaces

The probabilistic space generated by a grid on Figure 2a) is a pair (Ω_a, p_a), where $\Omega_a = \{F_1, F_2, F_3\}$ and $p_a(F_1) = \frac{1}{2}$, $p_a(F_2) = \frac{1}{8}$, $p_a(F_3) = \frac{3}{8}$.

3. Equivalence through splitting versus isomorphism of discrete probabilistic spaces

Definition 3-1 [ISOMORPHIC PROBABILISTIC SPACES] Probabilistic spaces (Ω_1, p_1) and (Ω_2, p_2) are *isomorphic* or *equivalent*, if there is a bijection g from the class Ω_1 to Ω_2 such, that

$$\forall \omega \in \Omega_1 \forall \varpi \in \Omega_2 : [\varpi = g(\omega) \Rightarrow p_2(\varpi) = p_1(\omega)].$$

We say that bijection g *determines isomorphism* and *saves probability.*

Definition 3-2 [FIGURES WITH GRIDS EQUIVALENT THROUGH SPLITTING] We say, that figures F with a grid $S_F = \{F_j : j \in S\}$ and G with a grid

$S_G = \{G_k : k \in S\}$ are equivalent through splitting if there is a bijection g from the class S_F to S_G such, that for every $j \in S$, if $G_j = g(F_j)$ the mesh G_j is an image of the mesh F_j in an isometry that is a rotation, a translation or a combination of both.

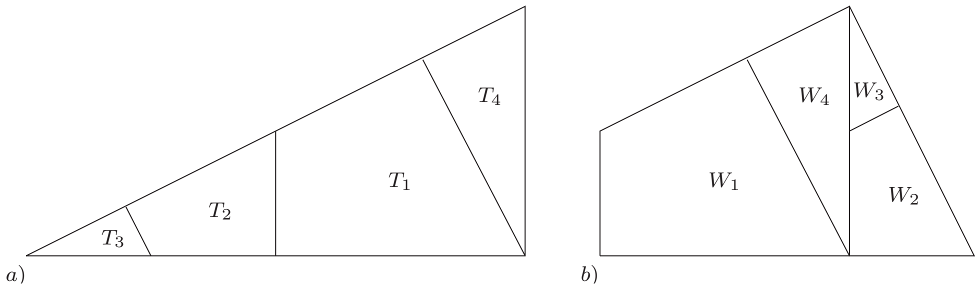


Fig. 3 A square K with a grid and a triangle T with a grid as figures equivalent through splitting

■ A square K with a grid of $\{K_1, K_2, K_3, K_4\}$ and a triangle T with a grid of $\{T_1, T_2, T_3, T_4\}$ from Figure **??** are figures equivalent through splitting. The bijection g is function $g(K_j) = T_j$ for $j = 1, 2, 3, 4$. The proof of that fact is a geometrical task. Its idea may be suggested by certain procedures done on those figures' models.

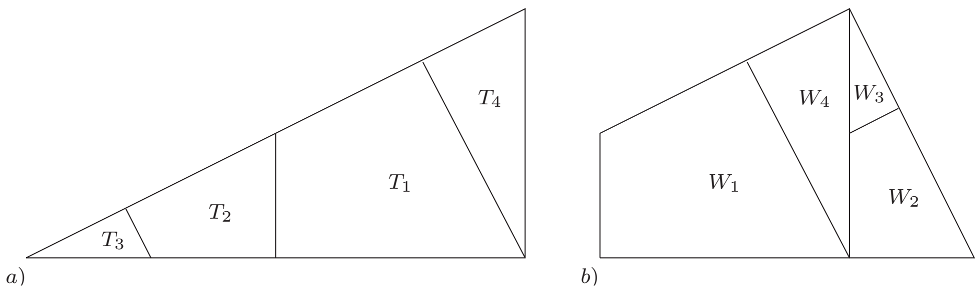
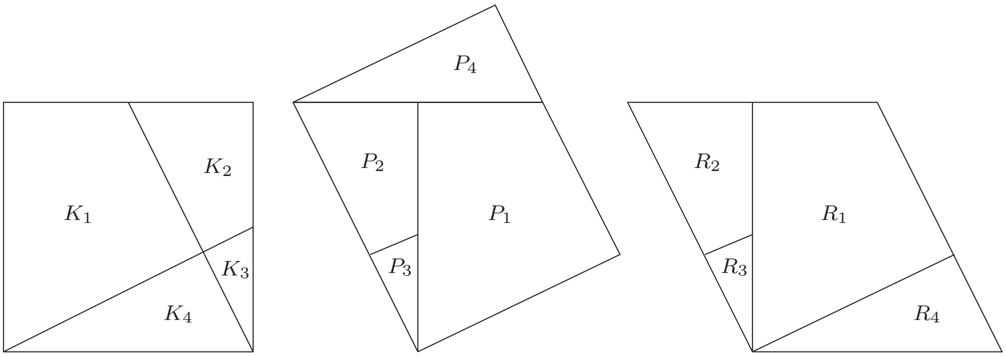


Fig. 4 Figures with grids equivalent through splitting

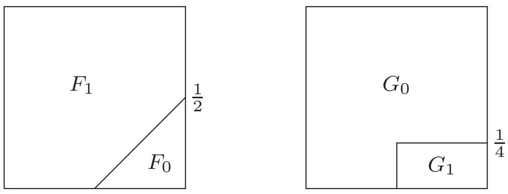
If two figures with grids are equivalent through splitting, the probabilistic spaces generated by grids imposed on those figures are isomorphic.



a) *b*) *c*)

Fig. 5 A square K, a rectangle P and a parallelogram R with grids as figures equivalent through splitting

■ Figure 5 shows three figures with grids, each two of them are equivalent through splitting. Each of the figures generates a discrete probabilistic space. Each two of those spaces are isomorphic. Defining each of them as a pair (Ω, p) is a geometrical task of calculating areas of certain figures.



a) *b*)

Fig. 6 Two isomorphic probabilistic spaces as geometrical objects

Two probabilistic spaces generated by grids on Figure 6 are isomorphic, but figures with those grids imposed are not equivalent through splitting. The bijection g, which defines the isomorphism is a function from the class $\{F_0, F_1\}$ onto the class $\{G_0, G_1\}$ determined by a formula $g(F_j) = G_{1-}$ for $j = 0, 1$.

4. A random discrete experiment and its probabilistic model – a model as a probabilistic space

□ [A RANDOM DISCRETE EXPERIMENT] A random discrete experiment is such an experiment (real or thought-of), that its effect is random, the set of results is not larger than countable and for each result we can *a priori* state or *a posteriori* estimate its probability (see [4], pp. 16–17 as well as [8], p. 13).

□ [A MODEL OF A RANDOM DISCRETE EXPERIMENT] The set of results of a random experiment has at least two elements and at most countable. A discrete probabilistic space of (Ω, p) is called *a model of a random discrete* *experiment* δ, if Ω is the class of all the possible results of the δ experiment and a function p associates every result the probability of the experiment δ ending with this result.

■ [A MODEL OF A COIN TOSS] Let us code the results of a coin toss with numbers: 0: `heads`, 1: `tails`. A model of this toss is a probabilistic space of (Ω_M, p_M), where

$$\Omega_M = \{0, 1\} \quad \text{and} \quad p_M(0) = p_M(1) = \tfrac{1}{2}.$$

■ [A MODEL OF THROWING A DICE] Let us code the results of throwing a dice with a number of stitches that show. A model of that throw is a probabilistic space of (Ω_K, p_K), where

$$\Omega_K = \{0, 1, 2, 3, 4, 5, 6\} \quad \text{and} \quad p_K(j) = \tfrac{1}{6} \text{ for } j = 1, 2, 3, 4, 5, 6.$$

A classical probabilistic space is a model of a random experiment only in case, when all the results of the experiment are equally possible.

5. A tangram of a discrete probabilistic space as its special geometrical presentation.

The probability associated to a result of a random experiment is, in a way, a kind of its measure, which can be interpreted (and visualized) as a figure's area.

□ [A TANGRAM OF A PROBABILISTIC SPACE] Let (Ω, p) be a non-singular discrete probabilistic space (so p should be a function with positive values). We interpret the elements of the Ω class as mashes of a grid imposed on a square with an area of 1 in such a way, that for every $\omega \in \Omega$ the number $p(\omega)$ is the area of the ω mash. The square with such a grid we shall call the tangram of the (Ω, p) probabilistic space.

If the (Ω, p) pair is a probabilistic space generated by a grid imposed on a square having the area of 1, that square with a grid is simultaneously the tangram of that space.

Every non-singular discrete probabilistic space (Ω, p) can be shown as its tangram. It is a measured (and so geometrical) presentation of that space (see [14]). In the same time it is a didactic way that allows visualization of such abstract probabilistic issues as an event and its probability.

■ [A TANGRAM OF A $K_{1\to 8}$ DICE THROW MODEL] The result of throwing a $K_{1\to 8}$ dice, which has numbers from 1 to 8 on its sides (its schedule is shown in Figure 7), is the number showing on the upper side after the dice falls.

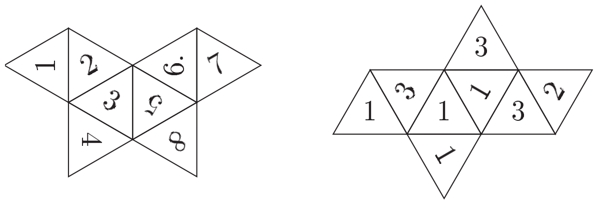


Fig. 7 Schedules of two eight-sided dice

The model of a $K_{1\to 8}$ dice throw is a probabilistic space (Ω_8, p_8), where

$$(\Omega_8 = \{1, 2, 3, 4, 5, 6, 7, 8\} \quad \text{and} \quad p_8(j) = \tfrac{1}{8} \text{ for } j \in \Omega_8.$$

Three tangrams of that classical probabilistic space (Ω_8, p_8) are shown in Figure 8.

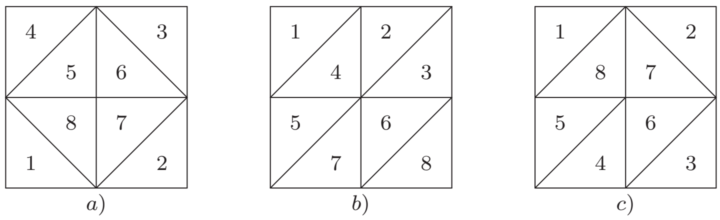


Fig. 8 Tangrams of a $K_{1\to 8}$ dice throw model

The model of throwing a $K_{11112333}$ dice, the schedule of which shows Figure 8*b*, is a probabilistic space of (Ω_8^*, p_8^*), where

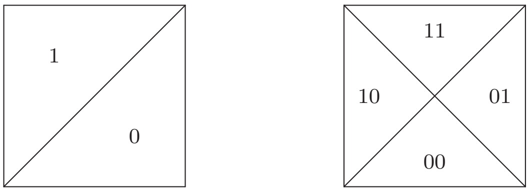
$$\Omega_8^* = \{1, 2, 3\} \quad \text{and} \quad p_8^*(1) = \tfrac{4}{8},\ p_8^*(2) = \tfrac{1}{8} \quad \text{and} \quad p_8^*(3) = \tfrac{3}{8}.$$

The probabilistic spaces (Ω_8^*, p_8^*) and (Ω_a, p_a) (the space generated by the grid shown in Fig. 8*a*) are isomorphic. The bijection that states the isomorphism is a function $g : \Omega_a \longrightarrow \Omega_8^*$, where $g(F_j) = j$ for $j = 1, 2, 3$.

Constructing of a tangram of a probabilistic space generated by a grid imposed on a positive-area figure may be accompanied by actual manipulations on models of the figure and the grid. Those real manipulations (similar to doing a jigsaw puzzle) may suggest all the imagined ones as parts of constructing a tangram as a mathematical object.

6. Models of random experiments with a coin and their tangrams

■ A tangram of a coin toss, i.e. an (Ω_M, p_M) probabilistic space, is shown in Figure 9*a*.

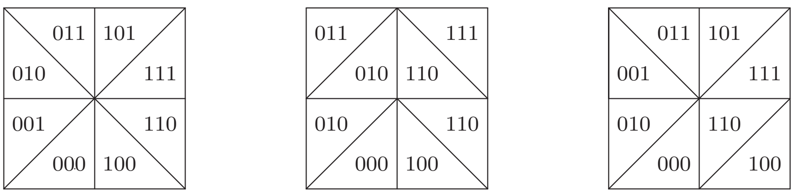


a) *b*)
Fig. 9 Tangrams of a single and double coin toss models

■ [A MODEL OF AN n-TUPLE COIN TOSS] The result of an n-tuple coin toss is an n-element variation of a $\{0,1\}$ class. Its j element is the result of the j^{th} toss. All the results of this experiment are equally probable (because in each single toss heads and tails are equally possible to show) and there are 2^n of them, so the model of an n-tuple coin toss is a probabilistic space of (Ω_{nM}, p_{nM}), where

$$\Omega_{nM} = \{0,1\}^n \text{ and } p_{nM}(\omega) = \frac{1}{2^n} \text{ for every } \omega \in \{0,1\}^n.$$

Note, that $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$, so $(\Omega_{nM}, p_{nM}) = (\Omega_M, p_M)^n$. The probabilistic model of an $n-$tuple coin toss is an n^{th} carthesian power of a single coin toss. A tangram of a double coin toss is shown in Figure 9*b*. Three tangrams of a triple coin toss are shown in Figure 10. Each of them is a square with a grid, that is equivalent through splitting to each of the tangrams in Figure 8. Apparently, we can simulate a triple coin toss with a single throw of a $K_{1\to 8}$ dice.



a) *b*) *c*)
Fig. 10 Tangrams of a triple coin toss

Let g be a function from the $\{1,2,3,4,5,6,7,8\}$ class (that is the set of possible results of a $K_{1\to 8}$ dice toss) onto the $\{0,1\}^3$ class (i.e. a set of results of a triple coin toss) that is stated as follows:

k :	1	2	3	4	5	6	7	8
g(k):	001	010	011	100	101	110	111	000

This function g defines the isomorphism of two probabilistic spaces: a model of a $K_{1\to 8}$ dice toss and a model of a triple coin toss. At the same time it is a „dictionary" to translate the result of a $K_{1\to 8}$ dice toss to the result of a triple coin toss. If the result of a $K_{1\to 8}$ dice toss is k, the result of a triple coin toss is $g(k)$.

A tangram of a probabilistic space, which is simultaneously a model of a multi-step experiment may be created in stages by dividing a unitary square. Let us assume, that vertical dividing lines mean odd stages of the experiment and horizontal dividing lines mean even stages. With this assumption the tangrams of single, double and triple coin toss look as it is shown in Figure 11.

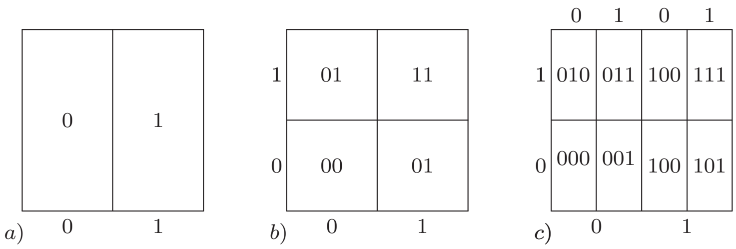


Fig. 11 Tangrams of single, double and triple coin toss

7. A tangram of a Cartesian product of probabilistic spaces – the probability of an event as an area of a figure

Figure 12 shows schedules of three dice: dodecahedronal K_{12}, hexahedronal K_6 and octahedronalK_8 .

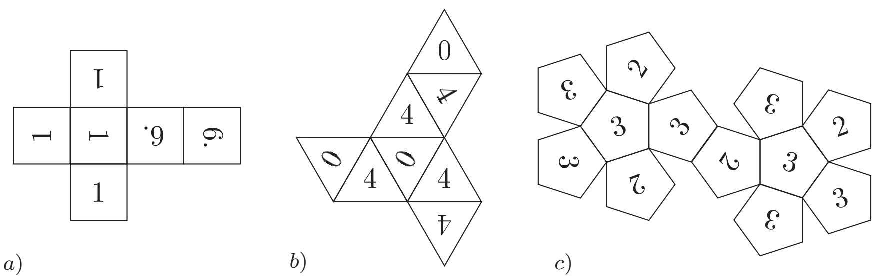


Fig. 12 The schedules of the three dice

A throw of each of them is a random experiment. Its result is a number that shows on the upper side of a dice when it falls. The K_6, K_{12} and K_8 dice are props in a lot game. There are two players, each has a dice. The one who gets a bigger number after this throw wins the game.

We will prove that in such a game the K_{12} dice gives a player a better chance to win than a K_6dice gives to his rival. The K_{12} dice is *better* than the K_6. We note this fact with a symbol $K_{12} \gg K_6$. We can see the arguments in Figure 13. It is about geometrical means of organizing the stages of counting.

Let us consider three probabilistic spaces:
— (Ω_{6-8}, p_{6-8}), which is a model of throwing the K_6 and the K_8 dice,
— $(\Omega_{8-12}, p_{8-12})$, which is a model of throwing the $K_8$and the K_{12} dice,
— $(\Omega_{12-6}, p_{12-6})$, which is a model of throwing the K_{12} and the K_6 dice.
Figure 13 shows a protocol of constructing the tangrams of those three probabilistic spaces. If the (Ω_j, p_j) pair models the throw of the K_j dice and the (Ω_{j-k}, p_{j-k}) pair models the throw of two dice: K_j and the K_l $(j = 6, 8, 12)$, then $(\Omega_{j-k}, p_{j-k}) = (\Omega_j, p_j) \times (\Omega_l, p_l)$.

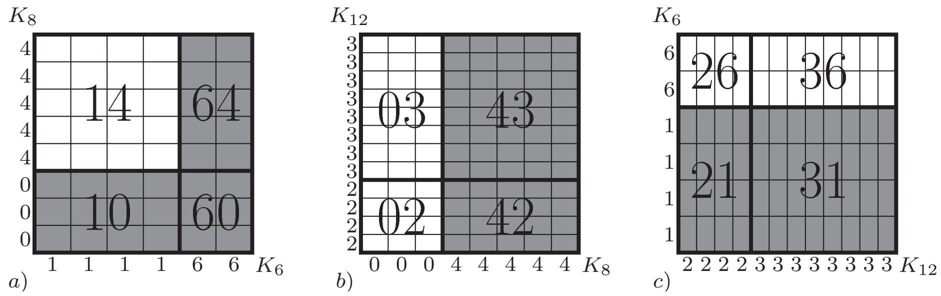


Fig. 13

Let us assume that in the game described above one of the players throws the K_6 dice, and his rival throws the K_8 dice. The blackened part (i.e. the set of blackened meshes of the tangram) in Figure 13*a* shows the event

$A = \{$*the* K_6 *dice shows a bigger number then the* K_8 *dice*$\}$.

The probability of this event happening is the area of the blackened part.

The white part (the set of the white meshes of the tangram) in Figure 13*a* shows the event

$B = \{$*the* K_8 *dice shows a bigger number then the* K_6 *dice*$\}$.

The area of the blackened part is bigger then the area of the white one, so $P(A) > P(B)$, that is the player who throws the K_6dice has in this game a better chance to win than his rival having the K_8 dice. And so it is $K_6 \gg K_8$.

Figure 13*b* shows, that $K_8 \gg K_{12}$ and Figure 13*c* shows that $K_{12} \gg K_6$. In case of the K_6, K_8 and K_{12} dice

$$K_6 \gg K_8 \quad \text{and} \quad K_8 \gg K_{12} \quad \text{and} \quad K_{12} \gg K_6.$$

So the relation of $\gg$ is not transitive in the set of $\{K_6, K_8, K_{12}\}$ dice. Among those dice there is no „best" one. For each of them there is a „better" one between the two others.

In this reasoning a tangram represents a probabilistic space, a set of meshes – some geometrical figure – is an event, and the area represents the probability of the event happening.

Intransitivity of the $\gg$ relation in the set of $\{K_6, K_8, K_{12}\}$ dice is a kind of a paradox. However, this fact may be easily transferred to the basis of reality. Let us assume, that the player was offered to choose his dice first. His rival would choose his dice from the two left. The intransitivity of the $\gg$ relation causes the decision of taking advantage of the right of priority is not a rational one.

Epilogue

This work shows:
– how to inspire translation of mathematical contents from the symbolic to iconic language and vice-versa;
– how to include geometrical means into stochastic reasoning;
– how to reduce counting probabilities to counting areas of figures.

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