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SOME REMARKS ON DEFINITION OF THE ABSOLUTE VALUE OF A REAL NUMBER

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Abstract. The article presents a didactic proposition of introducing the definition of the absolute value of a real number.

1. Introduction

This paper is a continuation of research into understanding of the absolute value of a real number. In the article (Major, Powazka, 2006) the necessary and sufficient conditions for the existence of the solutions of equations contained the absolute value functions were given. This paper contains the examples of didactic conceptions of the implementation of the definition of the absolute value of a real number at different levels of mathematical education. There were used functional equations of one or several variables. The didactic propositions described in this paper could be exploited by teachers of the secondary schools or the university teachers and mathematics students especially teaching oriented ones.

During their studies students learn different definitions of the absolute value of a real number. These definitions are based on the distance between two points on the number line, maximum of two real numbers or a square root of a nonnegative number. In the secondary school level one proves the following properties of the absolute value of a real number

$$|x \cdot y| = |x| \cdot |y|, \quad x, y \in R,$$

$$||x|| = |x|, \quad x \in R,$$

$$|ax| = a|x|, \quad x \in R, a \in R^+.$$

Let a function $\phi : R \to R$ satisfy the following equations

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y), \quad x, y \in R, \tag{1}$$

$$\phi(\phi(x)) = \phi(x), \quad x \in R, \tag{2}$$

$$\phi(a \cdot x) = a \cdot \phi(x), \quad x \in R, a \in R^+. \tag{3}$$

In this paper we study the conditions for which a solution of the system of equations (1), (2), (3) is in the form

$$\phi(x) = |x|, \quad x \in R. \tag{4}$$

2. Main results

In this part we prove four theorems. All of them could be use as a didactic proposition of introducing of the definition of the absolute value of a real number.

Proposition 1

We start with the following theorem of Cauchy [1].

Lemma 1. *If a continuous function $h : R \to R$ is for all real x, y a solution of the Cauchy functional equation*

$$h(x + y) = h(x) + h(y), \tag{5}$$

then there exists a real number λ such that

$$h(x) = \lambda \cdot x \tag{6}$$

for all real x.

Now we prove the following theorem.

Theorem 1. *If a nonconstant and continuous function $\phi : R \to R^+ \cup \{0\}$ is a solution of equation (1) for all real x, y with $\phi(a) = a$, where a denote a constant number and $a \in R^+ \setminus \{1\}$, then ϕ is the absolute value function (4).*

Proof. Let a nonconstant and continuous function $\phi : R \to R$ satisfy equation (1). Putting in (1), $x = y = 0$ we get $\phi(0) = \phi^2(0) \Leftrightarrow (\phi(0) = 0$ or $\phi(0) = 1)$. If $\phi(0) = 1$, then $\phi(x) = 1$ for all real number x, which is impossible, because ϕ is a nonconstant function. Hence we have

$$\phi(0) = 0. \tag{7}$$

Let x, y be positive real numbers. The substitution $x = e^u, y = e^v$, where $u, v \in R$ transform (1) into

$$\phi(e^{(u+v)}) = \phi(e^u) \cdot \phi(e^v). \tag{8}$$

Putting in (8) $g(u) := \phi(e^u)$ where $u \in R$. We get equation

$$g(u + v) = g(u) \cdot g(v). \tag{9}$$

Because g is a positive function, we have

$$\ln(g(u + v)) = \ln(g(u)) + \ln(g(v)). \tag{10}$$

Let $h : R \to R$ by the function given by $h(u) = \ln(g(u)), u \in R$. Then it follows from (10), that h is a solution of the Cauchy functional equation (5). By the definitions of functions g and h we get

$$h(u) = \ln(\phi(e^u)), \quad u \in R. \tag{11}$$

The continuity of the function (11) follows from the continuity the function ϕ and the logarithm function or the exponential function. It implies, that the function (11) is continuous solution of the Cauchy functional equation (5). In view of (6) and (11) there exists a number λ a such that

$$\phi(x) = e^{\lambda \cdot \ln x}, \quad x \in R^+,$$

thus

$$\phi(x) = x^{\lambda} m \quad x \in R^+. \tag{12}$$

Putting in (1) $x = y = t$ or $x = y = -t$, where $t \neq 0$ we have $\phi(t^2) = \phi^2(t)$ or $\phi(t^2) = \phi^2(-t)$, respectively. It follows that $\phi^2(t) = \phi^2(-t)$. By the assumption of Theorem 1 the function ϕ is a nonnegative solution of equation (1). Therefore

$$\phi(t) = \phi(-t), \quad t \in R \backslash \{0\}. \tag{13}$$

From (7), (12), (13) we get that continuous, nonnegative solution of equation (1) is given by

$$\phi(x) = |x|^{\lambda}, \quad x \in R. \tag{14}$$

Since a is a positive fixed point of the function ϕ, we get from (14), $a = \phi(a) = |a|^\lambda = a^\lambda$. From this we get

$$\lambda = 1. \tag{15}$$

From (14) and (15) it follows that the solution ϕ of the functional equation (1) is in the form (4).

Remark 1. *If we replace in Theorem 1 the continuity assumption by*

(i) ϕ *is continuous at a point,*

(ii) ϕ *is bounded from above on an interval,*

then the Theorem 1 holds true.

Proposition 2

The following results is generalization of Theorem 1.

Theorem 2. *If a nonconstant and continuous function* $\phi : R \to R$ *with*

$$\phi(a) = \phi(-a) = a, \tag{16}$$

where a *denote a constant real number and* $a \in R^+ \setminus \{1\}$*, is a solution of equation (1) for all real* x, y *then it is the absolute value function (4).*

Proof. Let $\phi : R \to R$ satisfy the assumption of this theorem. Similarly as in the proof of Theorem 1 we have (12). Since $\phi(a) = a, a \in R^+ \setminus \{1\}$, we get $\lambda = 1$. From this it follows that

$$\phi(x) = x, \quad x \in R^+. \tag{17}$$

Now, putting in (1) $x = y = t$ or $x = y = -t, t \neq 0$ we have $\phi(t^2) = \phi^2(t)$ or $\phi(t^2) = \phi^2(-t)$, respectively. Thus $\phi^2(t) = \phi^2(-t)$. Hence $\phi(t) = \phi(-t)$ or $\phi(t) = -\phi(-t)$. This and (16) yield that the function ϕ is an even function in R. Hence by virtue of (17) the function ϕ is the absolute value function.

Proposition 3

In this part we will show the method of defining the function given by (4), using a solution equation (2). We start with following.

Lemma 2. *If the function $\phi : R \to R$ have the inverse function, then the function (6) with $\lambda = 1$ is a solution of equation (2).*

Theorem 3. *If an even function $\phi : R \to R^+ \cup \{0\}$ is a solution of equation (2) and the restriction of this function to the interval $[0, +\infty)$ have the inverse function, then ϕ is given by the formula (4).*

Dowód. Let an even function $\phi : R \to R^+ \cup \{0\}$ satisfy equation (2) and the assumption of this theorem. Then $\phi(x) = \phi^{-1}(\phi(x)), \quad x \geq 0$, where ϕ^{-1} is the inverse function of ϕ. Hence we get $\phi(x) = x, \quad x \geq 0$. Because ϕ is an even function, formula (4) holds. □

Proposition 4

Now we will define the absolute value function using the solutions of the functional equation (3).

Theorem 4. *If an even function $\phi : R \to R$ satisfy the condition*

$$\phi(1) = 1, \tag{18}$$

is the solution of equation (3), then ϕ is given by the formula (4).

Proof. Putting $x = 1$ in (3) we have $\phi(a \cdot 1) = a \cdot \phi(1)$, where a is a positive real number. From this and (18) it follows that

$$\phi(a) = a, \quad a > 0.$$

Putting in (3) $x = 0$ we have $\phi(0) = \phi(a \cdot 0) = a \cdot \phi(0)$. Hence, $\phi(0) \cdot (1 - a) = 0$, $a > 0$. Because $a > 0$, we get that $\phi(0) = 0$. Therefore, we have the condition (17). Because ϕ is an even function, the formula (4) holds.

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications,* Academic Press, New York and London, 1966.

[2] J. Major, Z. Powązka, Uwagi dotyczące pojęcia wartości bezwzględnej liczby rzeczywistej, *Annales Academiae Paedagogicae Cracoviensis Studia ad Didacticam Mathematicae Pertinentia I*, Wydawnictwo Naukowe Akademii Pedagogicznej, Kraków, 163-185, 2006.