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ON OBJECTIVE AND SUBJECTIVE DIFFICULRTIES IN UNDERSTANDING THE NOTIONS OF THE LEAST UPPER BOUND AND THE GREATEST LOWER BOUNDS

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Abstract. The notion of the least upper bound (the greatest lower bound) of a subset of real numbers is discussed from different points of view and some difficulties of this notion are presented.

One can observe misunderstanding or insufficient understanding of the notion of the least upper bound (the greatest lower bound) of a set in the practice of school teaching. This notion is in fact very difficult but has a great importance in the modern mathematics. Let us start our considerations from some theoretical background. We will consider only least upper bounds, the greatest lower bounds one can define and discuss in the same way.

1. Theoretical foundations

In textbooks on mathematical analysis one can find different kinds of definition of least upper bound (supremum). Let us remind them.

Definition 1. A number d is called an upper bound of a nonempty subset A of the set of real numbers if

$$x \leq d$$

for each element x from the set A.

Definition 2. A number d is called the greatest element of a nonempty subset A of the set of real numbers, if $d \in A$ and $x \leq d$ for each element x from the set A.

Definition 3. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean the greatest number of this set or the least one from the set of all upper bounds.

The least upper bound of the set A is denoted by $\sup A$.

Definition 4. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound and

$$\big(\forall d' < d\big)\,(\exists x \in A)\,(d' < x \leq d) \tag{8}$$

Definition 5. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound of the set A and

$$(\forall \varepsilon > 0)\,\big(\exists x \in A\big)\,(d - \varepsilon < x \leq d) \tag{9}$$

Definition 6. By the least upper bound (supremum) of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound and

$$\big(\forall d'\big)\,(\forall x \in A)\,(x < d' \Longrightarrow d \leq d') \tag{10}$$

Definition 7. By the least upper bound of a nonempty subset A of the set of real numbers we mean a number d such that d is an upper bound of the set A and

$$\text{there exists a sequence } (a_n)_{n=1}^{\infty} \text{ of points of the set } A \text{ such that } \lim_{n\to\infty} a_n = d. \tag{11}$$

One can see that all definitions are equivalent for subsets of the set of real numbers. In fact. In each definition the least upper bound is an upper bound of the set A. So, to prove equivalences of those definitions we have to prove that each of the conditions (8), (9), (10) and (11) is equivalent to the statement that d is the least of all upper bounds of the set A.

Suppose that d is the least upper bound (in the meaning of definition 3. Now, if d' is any number less than d, then it is not an upper bound of the set A, hence there exists element x of the set A such that $d' < x$ (and of course $x \leq d$); in this way we have proved that d fulfils condition (8).

It is easy to see that conditions (8) and (9) are equivalent since the role of d' can play $d - \varepsilon$ and conversely, in place of ε one can take by $d - d'$ in adequate conditions.

To prove that definitions 4 i 6 are equivalent, let us notice that d' in condition (10) is an upper bound of the set A too, so it must be greater than d.

Conversely, if condition (10) is fulfilled, then no less element than d is an upper bound of the set A.

2. Commentary

At school programmes one can find the item *least upper bound (supremum)* of a subset A of the set $\mathbb{R}$ of real numbers. Many teachers prefer to introduce definition 3. This definition gives no suggestion how to check, whether an upper bound d is the least one. From this point of view the definition, which is very popular, is not very useful to apply. So it is no wonder that the students do not like to apply this definition and have several difficulties in it.

Next definition which is popular among the teachers is definition 7. This definition is much more difficult for students than definition 3. The most difficult problem in this definition lies in application the idea of limit of a sequence. So the two of most popular definitions have many disadvantages in applications and understanding of them.

In view of this remarks, the best condition for defining least upper bound of a set appears to be condition (8). It says that an upper bound d of a set A is the least upper bound if any less than d element is not an upper bound of the set A. For school purposes this definition seems to be the best one. So we can come to the conclusions, that the definition 4 should be the main one and after this definition had been introduced one can show the equivalence of all other conditions formulated in conditions (8) – (11).

3. Conclusions (Propositions)

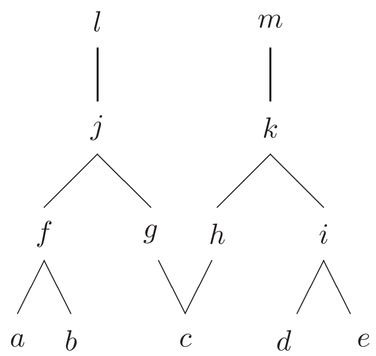
Let us consider now the problem of supremum of a set yet this time from the other point of view; I mean from the generalized kind of order. If we consider partially ordered set X, it means a set X equipped with relation of partial order. Let us remind; a relation $\prec$ is a partial order in a set X if it fulfils the following conditions:

- $x \prec x$,
- if $x \prec y$ and $y \prec x$ then $x = y$,
- if $x \prec y$ and $y \prec z$ then $x \prec z$,

for all elements x, y and z from the set X.

Nowadays we observe the enormous development of computer science. In many applications of this kind of science we can find the idea of (partial) order in small sets, namely finite sets. One can expect that such ideas will be found in school programmes quite soon. In partially ordered sets there is no possibility to define ideas like supremum in the way we discussed before. The problem lies in the fact that not every two elements of a set X can be comparable. Because of it we must define supremum in a way which had been pointed in definition 6. In consequence we have to define supremum by condition (10) since there is no other possibility to define it. Then the idea of partially ordered sets is the most general one. Later on, by specification, one can define supremum for subsets of a linearly ordered set; here one can use Definition 4. Definitions 5 and 7 can be useful only for subsets of the set of real numbers in which not only ordered structure but also algebraic (Definition 5) and topological structure (Definition 7).

There are some advantages for this sequence of conditions defining supremum. If X is a finite set with partial order in it, then one can illustrate it in a diagram. For example, if the set X is partially ordered in the way presented in the diagram:



then we can observe that:

$$\sup\{a,b\} = f, \qquad \sup\{a,b,c\} = j, \qquad \sup\{a,c\} = f$$

but there is no supremum of the set $\{a,d\}$.

Diagrams are very acceptable by students and improve the understanding of the very difficult idea of supremum. From this it is possible to come a little further and introduce the idea of Boolean algebra, which have many applications in computer sciences. I suppose that this kind of thinking is worth to discuss.

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