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ABOUT A DISTRIBUTION OF POINTS ON A LINE SEGMENT*

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1. Introduction

The concept of infinity is a key term in mathematics and its teaching. The development of knowledge about infinity makes an essential milestone important for its further development. If we accept the theory of so called genetic parallel that the ontogenetic development is not independent of the phylogenetic development, it is possible to assume that the obstacles we can identify in the phylogenetic development of infinity can be found also in ontogenesis and that overcoming the obstacles is a necessary component of the cognitive process of individuals.

The research activities conducted under the framework of the three year project GAČR *Obstacles in phylogenetic and ontogenetic development of the concept of infinity* are focused on the study of phylogeny development of infinity and also on the study of ontogenetic development of infinity among today's population. The research should be crowned by formulations of suggestions how to overcome the identified obstacles.

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2. Attributes of infinity

Infinity has many different aspects. Many mathematical concepts, methods, particular thoughts and developed theories are based on infinity. For the purpose of research, it was necessary first to untangle of mutually connected exposures and their attributes. Today, there are the following six attributes in the centre of our research interest:

- set cardinality (distinguishing finite, numerous and infinite sets, acceptance of actual and potential infinity);
- set boundedness (distinguishing bounded, very large and unbounded sets);
- set measure (geometrical measures point sets, images of infinitely small and large sets);
- ordering (perception of density of ordering, polarity of 'atomic' discrete images and images of continuum of number and geometric sets);
- infinite process (phenomenon of infinite repetition, induction, recurring);
- convergence (perception of the phenomena such as approaching, overcoming potential argumentation).

3. Obstacles in understanding infinity

An obstacle is understood as a set of mistakes related to previous knowledge. The knowledge can be successfully applied in certain situations but in a new context, its application leads to wrong solutions. To move further it is necessary to overcome the obstacle, to distinguish which of the pieces of knowledge can be transferred into the new context and which not and which pieces of knowledge appear to be new [3]. The research we conducted in this field brought out the following set of obstacles in understanding of infinity. We consider them to be principally irremovable (epistemological):

- experience with 'finiteness' (finite sets, finite processes, bounded objects, etc.);
- experience with the ordering of natural numbers;
- potential perception of the infinite process;
- replacement of an object with its model, especially with a picture;
- 'location of the horizon', which determines generally in a certain context the border between distinguishable and undistinguishable objects.

Besides the epistemological obstacles, we can also come across obstacles of different features we could overcome without disturbing the concept-making process. These obstacles can be cultural and didactical, for example, experience with colloquial language presenting variations of the word infinity in unacceptable mathematical contexts; and also the expectation that every task has a solution.

4. Research problem sample

The solution of the following problem presented by our respondents is central to our interest. *Given a square $ABCD$, find a point X on its side BC so that the triangle ABX will have: (a) as large its area as possible, (b) as small its area as possible.*

The majority of the respondents answer the first part of the problem well that it is necessary to place the point X onto C. For this reason, we focus on the second part of the problem.

Our assumption is that in the way of solving this problem, it will be possible to identify the extent of understanding of attributes such as the measure and the set ordering. We expect that pupils can face remaining obstacles closely connected to these attributes, caused by the transfer of the characteristics of a model (picture) into the object itself (into a point, a line, a triangle), by the experience with the ordering of the set of natural numbers (good ordering, isolated points) and also obstacles connected with the movement of the horizon and its tilting behind.

4.1. Expected answers

In the following analysis we will assume that the respondent is a primary level pupil, a secondary level student, or a university student. We assume that the students will consider the presented problem in classical Euclidean space as they (with exceptions) have never come across different types of space. We can rank the expected answers (starting with a poor image of infinity up to a sophisticated image) as follows:

- point X will be about 1mm (resp. any concrete 'minute' distance) from point B: here, the significant element is the picture as mentioned above, and also the **creator principal**, by which we consider such an approach of pupils and students that they think they have to create and construct the required object, find the point, and their effort is limited by their human senses and abilities;

- point X is right next to point B: considering the previous answer, there is a certain shift of the horizon. The respondent realizes that any distance can be diminished. However, the approach is the same as in the previous case as he/she expects an answer based on the construction of a picture;
- point X will be infinitely close to point B: it is an analogy to the previous answers, there is again a shift of the horizon (possible analogy with infinitely small quantities);
- the location of point X cannot be determined (for the reason of limited human possibilities): point X is placed on the very horizon, again the creator principal;
- the location of point X cannot be determined (thanks to the possibility to move point X from any place to point B): argumentation based on dynamic imaginary and potential approach;
- such a point does not exist: the reason can be either the image of potential approach as well as conscious work with lower bounded sets with no minimum, resp. with finite open set;
- point X will be in point B: acceptance of a line segment as a triangle with zero area influenced probably by the knowledge of limits.

4.2. Interview samples

The research conducted since 2005 has been focused on the images of secondary level students aged 15 to 19. The experiment question was given in a written form with the picture of square $ABCD$. The responses were recorded using a camcorder or voice-recorder.

Interview 1: Martina, 18 years

M46: *I will place it somewhere just right above B. However, I cannot determine where...*

E47: *And why not?*

M47: *It is immeasurable... it is possible to measure it... I can say that side BC has 1 cm, but if I want the area to be as small as possible, so if I place it just above B it is always possible to get a smaller area.*

M50: *Oh, it is... the smallest possible distance above B.*

Martina argues for the potential decrease of the distance between points X and B. In the end, she accepts the hypothesis of the smallest distance between two pints. It can be caused by experience with limits (when infinite processes are assigned with solid concrete values), as she could come across this concept in year 4 at her grammar school. Answer M50 can also be influenced by the fact that Martina expects some (positive) answer. Also the considered distance has been changed from żight above"in M46 and "just above"in

M47 to śmallest distance possible"in M50. We can consider the "bit"to be in front of the horizon - we can imagine it even though it is smaller and smaller - however, we cannot imagine "the smallest possible distance- it has tilted behind out horizon. This tilt behind the horizon is identified by Martina with the searched position of point X. The fall behind the horizon concerning the differentiability can be indicated also by the response "it is immeasurable there"in M47.

Interview 2: Vašek and Martin, 15 and 16 years

M15: *If it was totally closed, there could be a distance of one single undefined point.*

E16: *(2s) Well, but a point is not a distance.*

M16: *It is not, but if it is not defined, there must be a certain minimal diameter. (2s) There must be a tiny little gap...*

V17: *I think there is nothing.*

...

V18: *But the point will have probably some length, otherwise a set of points do not make a straight line.*

E19: *Well, so we have come across a fact that a line is a set of points. OK... (3s) However, Martin thinks there is something between the X and B.*

M19: *There must be something minimal, otherwise if the points are on each other it would be the triangle.*

E20: *And do you think that XB is a segment?*

M20: *It is a line segment with the length of a single point.*

Martin and Vašek consider a point to be the smallest measure - they have a pure atomic image. Although Vašek does not accept directly Martin's justification (V17), in the end he argues that something having a zero size could not make a straight line (V18). Vašek is brought to the conclusion of a zero point by the argumentation about actual infinite set. Vašek is not experienced enough and does not have sufficient mathematical skills to overcome the seeming contradiction. However, he is aware of it. His knowledge obtained in his existing education contradicts itself which can be deducted from answer V18: "but the point ... ąnd "probably". On the other hand, Martin comes to the conclusion probably thanks to his image of finite objects and with his experience with models - i.e. pictures etc. Like in the previous conversation, Martin places the point beyond the horizon of differentiability when he speaks about "undefined point"in M15 and M16. Again, he expects a smallest non-zero value - łength of one point"in M20.

The conversation samples are accompanied with pieces of conversation between younger pupils. The problem they should solve was formulated as follows: Given line segment AB, divide it in the ratio 1:2. What objects are created? Even though the question sounds different, its aim is to explain again images about points and their distribution on a line.

Interview 3: Markéta, Lenka and Jirka, 13 years

L14: *No, I will cut to pieces also the point. (Laugh)*

E15: *And is it possible to do it?*

J16: *It is nonsense! Simply, the point will be one and not on the other. It will be a half -line, a half line-segment.*

J18: *Let's put it on this one. (Leaving line segment AC, crossing out the end point C by the other.)*

E19: *OK, now we have no point C on the other, where does it end?*

J20: *Nowhere!*

L21: *It ends here! (Pointing to point C.) (1s) The point is not there. (1 s) It is here. (Pointing to line segment AC) (1s) Well, we can define another point... Here (pointing to point C) we will name it with D.*

L35: *As there is point C, and there is the line, it will be halved. It is for instance one millimetre, so we will halve it and here will be half a millimetre and there as well.*

Lenka works from the very beginning with a point as if it was a very small object - not as if it was a unit, as she accepts further division (L14, L35). It is mainly because of the fact that the point blends with the model - the drawn line on the line segment (L21, L35). She even chooses a size "for instance one millimetre"(L35). The narrow focus on the model is supported by the above mention creator principle. Lenka wants to exercise physically or she imagines that she would do everything she mentions in the conversation. All responds are evaluated considering the presenter. On the other hand, Jirka can approach the problem in an original way (considering his age) when he speaks about an object he has never heard of - about a half segment-line (J16), which is an analogy to a half-open interval. He can even think about consequences of the ópeningóf the segment-line when he says that the half segment-line ends ńowhere"(J20). How far can he go in his thoughts, we can only assume. Dominant Lenka did not leave enough room to Jirka to develop his idea. Jirka could not be as sure as Lenka, as he was just grasping it. On the other hand, Lenka worked with her object images which have been developed since the first meeting with geometry and for this reason she worked more self-confidently. The dominance of Lenka is presented also by the fact that Markéta was given no space to express her ideas and so she became an observer of Lenka's argumentations.

We summarize the obtained results flowing out from the conversations in the final paragraph. Before we do it, let's outline relevant realized pieces of research.

5. Connection with other research

Similar problems are examined in two pieces of recent research. The first one is the research focused on the understanding of a line segment as a set of points and with related ideas and the second is research dealing with the density of number sets, which is an analogy with our problem in arithmetic context.

5.1. About cardinality of sets of points

Bikner-Ahsbahs formulated in her research of 10 year old children, images of geometrical objects [4]. One of the questions of her questionnaire as follows: Wie viele Punkte liegen auf dieser Strecke? (How many points are there on this segment line?) In the questionnaire, there was a drawn line segment with a length of about 2 cm. Bikner-Ahsbahs presents that the majority of pupils declared the number of 30. In her research exercise, the pupils sketched points on the image of the line segment and then count them. After the incentive question Are there more points?, they responded in a way that tried to squeeze more points onto the segment line. After filling the whole line with points, they come to the conclusion, there are no more points.

These results reflect the fact that the majority of primary level children understand a point its by its image on paper or a blackboard. We can legitimately assume that this is an obstacle as the transfer of characteristics from the model, resp. from the picture onto the object itself, prevalent with primary level pupils and very often also with older students [6].

5.2. About density of number set arrangement

The second piece research was conducted by Eisenmann in the years 1998–2000 in 30 primary and secondary schools in the region of North Bohemia [5]. Among questionnaire items, there were also the following questions:

- Which number is the smallest bigger than 0? (year 1 - 5)
- Which is the smallest decimal number bigger than 0? (year 7)
- Which is the smallest rational number bigger than 0? (year 9)
- Which is the smallest real number bigger than 0? (year 11 - 13)

The gradual changes in the formulations of the question obtained in the pilot tests, where every question was rectified concerning its suitability for the particular age group are shown above. From the mathematical point of view, there are two phenomena - set minimum and the density of ordering. If we detach our attention from year 1, 3 and 5; we deal with fact that the set of decimal, rational, and real numbers bigger than 0 have not the minimum but only the infimum. The reason is that these sets of numbers are densely arranged.

It is natural that the most common answer in years 1, 3, and 5 is the answer 1. The pupils take into consideration only the set of natural numbers. The rest of the answers in years 1 consists above all the answer I do not know (23%) and incorrect answers 3, 10 (10%) or 0 (8%). In years 3 and 5, there also answers such as 0.1, 0.01, 0.00...01 or 0.000...1. The answers such as 0.00...01 are the most common in each age group (starting with year 7).

In year 5, there are also sporadic answers such as 0.1, 0.0001 or $\frac{1}{2}$, $\frac{1}{3}$. The uniqueness of the last answers is natural. Pupils in year 4 are introduced to fractions.

However, they do not consider them to be numbers but as operators or a part of a unity. From the phylogenetic point of view, the concept of number corresponds with Euclid and his Fifth Book of The Foundations, as there is the ratio 2:3, a quantity relation between two concrete quantities (see [10]). One of the first situations, when pupils can start to perceive fractions as numbers, is their representation on the number line. And this representation is a suitable means for later understanding of the fact that the set of all rational and real numbers are densely arranged. Early in year 4, subtraction of bigger natural numbers is illustrated using the number line and the principle of a magnifying glass. If teachers use this principle also later on when talking about decimal numbers, pupils acquire better images about the density of the given set of numbers.

Let's talk about students' results of the last year (13) of their secondary education. In this group, there are 13% of respondents who state that the lowest real number bigger than 0 is number 1. Interviews with students showed that these individuals ignored in the given question the word real, under the word number they automatically see an integer or a natural number.

In this year group, the respondents used for the first time the answer $\frac{1}{\infty}$ (with a frequency of 20%), and they also used quite often (11%) the equivalent answers None and Does not exist. We interviewed the majority of the respondents answering the above mentioned way to find out the explanation and the reasons for their answers. With only one exception, all students explained meaningfully their correct answers. We can always find a smaller and smaller number; it will never be the smallest (Honza). We can make a half of the small number which is always smaller, (3s) it will never be zero (Jakub). So I can add another zero (in number 0.0000001 - author's note), and so on, it will never end with a smallest number (Hedvika).

An interesting parallel of the discussed problem is illustrated by the following chunk of conversation between two students of year 13, which took place an hour after the filling in of the questionnaire. Experimenter: *Jana, you have answered in this questionnaire* 0.000000000001 *to the question "What is the smallest real number bigger than 0?".*

Jana1: *Yes.*

E2: *Karel, do you think that it is right? (Karel wrote Does not exist.)*

Karel2: *No, for instance (writing on paper) 0.000000000000001 is smaller.*

E3: *(to Jana) Well, (2s) is he right?*

J3: *Hm, yes, he is...*

E4: *And what is then the smallest real number bigger than 0?*

J4: *(4s)*

K4: *None. I can always find smaller. (3s) Look. (Sketching the real number line on paper, marking 0 and point $\frac{1}{2}$.) I will halve this (marking $\frac{1}{4}$), and again this (marking $\frac{1}{8}$) and so on (sketching other five short lines towards zero). I can divide it forever and never reach the smallest number.*

J5: *Then, there must be some infinite small line segments. (3s)*

K5: *(Surprisingly) Well, (4s) well, but (2s) you understand, the right end point is the number, and I can always make it smaller, and so no smallest one exists.*

J6: *Well, but a line segment yes - we had in class mathematics a week ago when we talked about a circle (3s) its length (2s) and how to make its circumference by* *inscribing the polygons with bigger and bigger number of vertices (2s) up to the one with an infinite number of line segments (2s) or vertices (2s) which is the circle. Therefore, the infinitely short line segment exists.*

Jana aims, in her argumentation, at a different direction than Karel who keeps the idea of the smallest set element. Jana does not speak about the smallest but about an infinitely small line segment. It is not an analogy of the smallest real number bigger than zero, but about an analogy of an infinitely small quantity.

6. Conclusions and connections with the historical development of images of infinity

The chosen question enables us to distinguish sufficiently and in detail the level of understanding points of and their distribution on a line in connection with the attributes of infinity. The respondents have a natural possibility of justifying their answers. Thanks to this fact, we can confirm the presence of the obstacles.

We can see the importance of the horizon. By the movement of the horizon and its crossing, the respondents make their answers more precise. At the same time when they think beyond the horizon - as far as they can see they cannot do without a kind of indeterminateness - "undefined pointór "immeasurableąnd so on. The awareness of the horizon and the world beyond the horizon stands at the beginning of our European science. The need of crossing the horizon accompanied by fear and uncertainty of its crossing is evident in the history of Euclid's Fifth Postulate. Why did so many people want to "prove"it? (Even after it had been proved that this postulate is independent of the other axioms.) It is because of the fact that the truth about the world in front of the horizon is grounded about something beyond the horizon [8].

All mentioned conversations have a close connection with the atomism of Democritus. However, each of the respondents is in a different phase of understanding a point as an atom. It is apparent in interview 2. While Martin has a naive image of an atom to be a sufficiently small object, Vašek comes to his conclusion after his consideration of a line as a set of points. It is an essential parallel with Democritus' theory formulated according to the paradoxes of Zenon based on the dispute between potential and actual infinity. In this case, it is based on the paradox that the sum of infinitely small quantities can be a non-zero value [1, 7]. Aristotle formulates the basic principle "what does not have a size that does not add a size"[1].

The idea of atomism is not only typical of Democritus. Also Descartes, who was a founder of analytical geometry, uses in his work *Principles of Philosophy* 'next-bordered' points, when he claims that two touching spheres have two points of contact, each of them one [9]. It refers to the fact that the match of a straight line and the number line is an enormously difficult thought construct.

There is an obvious impact of a model, especially a picture, on the process of image creation of geometrical objects supported by the creator principle. These phenomena can be traced in the history of mathematics from the beginning of mathematics development to recent history. The criterion of existence has been changing. Publisher of Bolzano still respects the proof of the mean value theorem, but he adds: "It is obvious that such a function does not exist"[2], as it is not even possible to imagine it [9]. The analogy can be found in the late acceptance of the existential quantifier.

We assume that the further phase of our research, focused on extended quantitative and qualitative investigation, will bring more solid results. We plan to conduct more detailed analyses of interviews and to extend the problems' scope to cover all attributes of infinity and to open the possibility of identification of other obstacles.

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