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RECURRENT EQUATIONS FOR THE ARITHMETICAL AND GEOMETRICAL SEQUENCES OF HIGHER DEGREE

Arkadiusz Bryll[a], **Grzegorz Bryll**[b], **Robert Sochacki**[b]

[a] *Technical University of Częstochowa*
Dąbrowskiego 69, 42-200 Częstochowa, Poland

[b] *Institute of Mathematics and Informatics*
Opole University
ul. Oleska 48, 45-052 Opole, Poland
e-mail: sochacki@math.uni.opole.pl

Abstract. Recurrent equations concern relationships between some (in general in a neighbourhood) elements of sequences. By these equations one can evaluate an arbitrary element of such sequences. In this paper we consider recurrent equations for the arithmetical and geometrical sequences of higher degree. We also give some properties of these sequences.

Let us start with a short introduction to definitions and properties of the sequences of higher degree. For the given sequence $\{a_n\}$ we define the sequence of the m-th differences $\{\Delta^m a_n\}$ and the sequence of the m-th quotients $\{q_{n,m}\}$ in the following way:

$$\begin{cases} \Delta^1 a_n = a_{n+1} - a_n, \\ \Delta^{m+1} a_n = \Delta^m a_{n+1} - \Delta^m a_n, \quad n, m \in N \setminus \{0\}. \end{cases} \tag{1}$$

$$\begin{cases} q_{n,1} = \frac{a_{n+1}}{a_n}, \\ q_{n,m+1} = \frac{q_{n+1,m}}{q_{n,m}}, \quad n, m \in N \setminus \{0\}. \end{cases} \tag{2}$$

The following two tables show elements of the considered sequences:

$$
\begin{array}{c|ccccc}
a_1 & & & & & \\
 & \Delta a_1 & & & & \\
a_2 & & \Delta^2 a_1 & & & \\
 & \Delta a_2 & & \Delta^3 a_3 & & \\
a_3 & & \Delta^2 a_2 & & & \\
 & \Delta a_3 & & \Delta^3 a_2 & & \ddots \\
a_4 & & \Delta^2 a_3 & & \vdots & \\
 & \Delta a_4 & & \vdots & & \\
a_5 & & \vdots & & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
$$

$$
\begin{array}{c|ccccc}
a_1 & & & & & \\
 & q_{11} & & & & \\
a_2 & & q_{12} & & & \\
 & q_{21} & & q_{13} & & \\
a_3 & & q_{22} & & q_{14} & \\
 & q_{31} & & q_{23} & & \ddots \\
a_4 & & q_{32} & & \vdots & \\
 & q_{41} & & \vdots & & \\
 & \vdots & & & & \\
\vdots & & & & &
\end{array}
$$

The arithmetical and geometrical sequences of higher degree are defined as follows (see [1-3]):

Definition 1. The sequence $\{a_n\}$ is called the arithmetical sequence of the k-th degree ($k = 1, 2, 3, \dots$) if and only if the sequence $\{\Delta^k a_n\}$ is constant and $\{\Delta^k a_n\} \neq 0$. Any constant sequence is called arithmetical sequence of the 0-th degree.

Definition 2. The sequence $\{a_n\}$ is called the geometrical sequence of the k-th degree ($k = 1, 2, 3, \dots$) if and only if the sequence $q_{n,k}$ is constant and $q_{n,k} \neq 1$. Any constant sequence is called geometrical sequence of the 0-th degree.

Theorem 1. Any arithmetical sequence $\{a_n\}$ of the k-th degree has the following properties:

a) $a_n = \binom{n-1}{0} a_1 + \sum_{i=1}^{k} \binom{n-1}{i} \Delta^i a_1$,

b) $s_n = \sum_{i=1}^{n} a_i = \binom{n}{1} a_1 + \sum_{i=1}^{k} \binom{n}{i+1} \Delta^i a_1$,

c) $\Delta^1 a_n = a_{n+1} - a_n = \sum_{i=1}^{k} \binom{n-1}{i-1} \Delta^i a_1$.

Theorem 2. Any geometrical sequence $\{a_n\}$ of the k-th degree has the following properties:

a) $a_n = a_1 \cdot \prod_{i=1}^{k} q_{1,i}^{\binom{n-1}{i}}$,

b) $\pi_n = \prod_{i=1}^{n} a_i = a_1^n \cdot \prod_{i=1}^{k} q_{1,i}^{\binom{n}{i+1}}$,

c) $q_{n,1} = \frac{a_{n+1}}{a_n} = \prod_{i=1}^{k} q_{1,i}^{\binom{n-1}{i-1}}$.

Theorem 3. For any sequence $\{a_n\}$, the general formulas of the sequences $\{\Delta^m a_n\}$ and $\{q_{n,m}\}$ given by (1) and (2), respectively, have the following forms:

$$\Delta^m a_n = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{n+m-i}, \tag{3}$$

$$q_{n,m} = \prod_{i=0}^{m} a_{n+m-1}^{(-1)^i \binom{m}{i}}. \tag{4}$$

Proof of the formula (3). The proof is by induction on the parameter m (for every $n \in N_+$). Let $m = 1$. Then the formula (3) is true, since $\Delta^1 a_n = a_{n+1} - a_n = (-1)^0 \binom{1}{0} a_{n+1} + (-1)^1 \binom{1}{1} a_n$. Let us assume now that formula (3) holds for $m = k$. By inductive assumption we obtain $\Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{n+1+k-i} - \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{n+k-i} = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}$. So, the formula (3) holds for $m = k+1$. By the principle of mathematical induction we conclude that the formula holds for every natural number $m \in N_+$.

Proof of the formula (4). Let $m = 1$. Then the formula (4) holds, since $q_{n,1} = \frac{a_{n+1}}{a_n} = a_{n+1}^{(-1)^0 \binom{1}{0}} \cdot a_n^{(-1)^1 \binom{1}{1}}$. Assume now that (4) holds for $m = k$. Using the induction hypothesis the expression $q_{n,k+1}$ can be rewritten as:

$$q_{n,k+1} = \frac{q_{n+1,k}}{q_{n,k}} = \frac{\prod_{i=0}^{k} a_{n+1+k-i}^{(-1)^i \binom{k}{i}}}{\prod_{i=0}^{k} a_{n+k-i}^{(-1)^i \binom{k}{i}}} = \prod_{i=0}^{k+1} a_{n+k+1-i}^{(-1)^i \binom{k+1}{i}}.$$

Thus the formula holds for $m = k+1$. So, by the principle of induction we can conclude that the formula (4) holds for every $m \in N \setminus \{0\}$.

Theorem 4.

a) If the sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree, then the recurrent equation

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}a_{n+k+1-i}=0 \tag{5}$$

is satisfied by this sequence.

b) If the sequence $\{a_n\}$ is the geometrical sequence of the k-th degree, then it satisfies the following recurrent equation:

$$\prod_{i=0}^{k+1}a_{n+k+1-i}^{(-1)^i\binom{k+1}{i}}=1. \tag{6}$$

Easy proofs will be omitted.

Theorem 5. The sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree if and only if the sequence $\{u_n\}$ of the form $u_n = A\cdot r^{a_n}$ (where $r\in R_+\setminus\{1\}$ and $A\neq 0$) is the geometrical sequence of the k-th degree.

Proof. On the basis of Theorem 3 (formulas (3) and (4)) we have:

$$q_{n,k}=\frac{q_{n+1,k-1}}{q_{n,k-1}}=\frac{\prod_{i=0}^{k-1}u_{n+1+k-1-i}^{(-1)^i\binom{k-1}{i}}}{\prod_{i=0}^{k-1}u_{n+k-1-i}^{(-1)^i\binom{k-1}{i}}}=\frac{\prod_{i=0}^{k-1}(Ar^{a_{n+1+k-1-i}})^{(-1)^i\binom{k-1}{i}}}{\prod_{i=0}^{k-1}(Ar^{a_{n+k-1-i}})^{(-1)^i\binom{k-1}{i}}}=$$

$$=\frac{A^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}}\cdot r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+1+k-1-i}}}{A^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}}\cdot r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+k-1-i}}}=$$

$$=r^{\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+1+k-1-i}-\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}a_{n+k-1-i}}=$$

$$=r^{\Delta^{k-1}a_{n+1}-\Delta^{k-1}a_n}=r^{\Delta^k a_n}.$$

Hence, it is shown, that $q_{n,k}=const\neq 1$ iff $\Delta^k a_n=const\neq 0$. Therefore, by Definitions 1 and 2 it is obvious, that Theorem 5 holds.

Corollary 1. If the sequence $\{a_n\}$ is the arithmetical sequence of the kth degree, then the sequence defined by the formula $u_n = Ar^{a_n}$ ($A\neq 0$, $r\in R_+\setminus\{1\}$) satisfies the equation (6).

It follows from Theorems 5 and 4b.

In order to solve the equation (5) we use the theory of homogeneous linear recurrent equations. Recall, that the recurrent equation of order k is of the form:

$$u_{n+k} = d_1 u_{n+k-1} + d_2 u_{n+k-2} + \ldots + d_{k-1} u_{n+1} + d_k u_n \ , \quad (d_k \neq 0) \tag{7}$$

has the following characteristic equation:

$$r^k = d_1 r^{k-1} + d_2 r^{k-2} + \ldots + d_{k-1} r + d_k. \tag{8}$$

Theorem 6 (see [4]).

a) If the characteristic equation (8) has k distinct roots $u_n^{(i)} = r_i^n$, then the general solution of (7) has the form:

$$u_n = \sum_{i=1}^{k} c_i r_i^n, \tag{9}$$

where the coefficients $c_1, c_2, \ldots, c_k$ are constants. One can see, that the general solution (9) is linear combination of particular solutions of the equation (7): $u_n^{(1)} = r_1^n$, $u_n^{(2)} = r_2^n$, $\ldots$, $u_n^{(k)} = r_k^n$.

b) If the characteristic equation (8) has distinct roots $r_1, r_2, \ldots, r_t$, $(t \leq k)$, of multiplicities $k_1, k_2, \ldots, k_t$, where $\sum_{i=1}^{t} k_i = k$, then the characteristic polynomial $\varphi(r) = r^k - (d_1 r^{k-1} + d_2 r^{k-2} + \ldots + d_{k-1} r + d_k)$ can be factored as: $\varphi(r) = (r - r_1)^{k_1} \cdot (r - r_2)^{k_2} \cdot \ldots \cdot (r - r_t)^{k_t}$. Then, the general solution of the equation (7) has the form:

$$u_n = \sum_{i=1}^{t} r_i^n \cdot \sum_{j=0}^{k_i - 1} c_{ij} n^j, \tag{10}$$

where c_{ij} $(i = 1, \ldots, t,\ j = 0, 1, \ldots, k_i - 1)$ are an arbitrary constants.

After expanding the formula (10) will be of the form:

$$\begin{gathered} u_n = r_1^n (c_{1,0} n^0 + c_{1,1} n^1 + \ldots + c_{1,k_1-1} n^{k_1-1}) + \\ + r_2^n (c_{2,0} n^0 + c_{2,1} n^1 + \ldots + c_{2,k_2-1} n^{k_2-1}) + \\ + \ldots + r_t^n (c_{t,0} n^0 + c_{t,1} n^1 + \ldots + c_{t,k_t-1} n^{k_t-1}). \end{gathered}$$

Any particular solution of (7) can be obtained from the general solution by suitable choice of constants. These constants are determined by the initial conditions.

Let the formula (5) has the expanded form:

$$(-1)^0\binom{k+1}{0}a_{n+k+1}+(-1)^1\binom{k+1}{1}a_{n+k}+\ldots+$$

$$+(-1)^k\binom{k+1}{k}a_{n+1}+(-1)^{k+1}\binom{k+1}{k+1}a_n=0. \tag{11}$$

Corollary 2. The general solution of (11) has the form:

$$a_n = c_0n^0 + c_1n^1 + \ldots + c_kn^k. \tag{12}$$

Proof. The equation (11) has the following characteristic equation:

$$(-1)^0\binom{k+1}{0}r^{k+1}+(-1)^1\binom{k+1}{1}r^{k}+\ldots+$$

$$+(-1)^k\binom{k+1}{k}r+(-1)^{k+1}\binom{k+1}{k+1}=0. \tag{13}$$

It is easy to see, that the left side of (13) is the expansion of the formula $(r-1)^{k+1}$ by the Newton's binomial. Thus $(r-1)^{k+1}=0$ if $r=1$. So, the characteristic equation (13) has the only one root $r_1=1$ with multiplicity $k+1$. Theorem (6) (the formula (10)) states, that the general solution of (11) is the sequence $\{a_n\}$ given by (12).

Example. Find a closed-form expression for the sequence
$$\{a_n\} = (0, 0, 2, 6, 12, 20, 30, 42, \ldots).$$

The general formula of the sequence $\{a_n\}$ is determined by Theorem 4a and Corollary 1. The sequence $\{a_n\}$ is the arithmetical sequence of the 2-nd degree and its recurrent equation has the following characteristic equation (see (5) and (11) for $k=2$): $(r-1)^3$. The only root of that is the number 1 with multiplicity 3. Using the formula (12) we obtain $a_n = c_0n^0 + c_1n^1 + c_2n^2$, i.e. $a_n = c_0 + c_1n + c_2n^2$. The constants c_0, c_1, c_2 are determined by the initial conditions: $a_1 = 0$, $a_2 = 0$, $a_3 = 2$. Then, we have:

$$\begin{cases} a_1 = 0 = c_0 + c_1 + c_2 \\ a_2 = 0 = c_0 + 2c_1 + 4c_2 \\ a_3 = 2 = c_0 + 3c_1 + 9c_2 \end{cases}$$

The solutions are: $c_0 = 2$, $c_1 = -3$, $c_2 = 1$. Finally, the given sequence is of the form: $a_n = 2 - 3n + n^2$. We can also use Theorem 1a to find the general formula of that sequence.

Theorem 7. The sequence $\{a_n\}$ is the arithmetical sequence of the kth degree if and only if its general formula is a polynomial of degree k.

Proof. Let $\{a_n\}$ be an arithmetical sequence of the kth degree. This sequence fulfils (11). By Corollary 2 we obtain $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$. By assumption and Definition 1 we have the following condition: $\Delta^k a_n = const \neq 0$, which implies $c_k \neq 0$, what means that the polynomial $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$ is of degree k.

Now, let us assume that $a_n = c_0 n^0 + c_1 n^1 + \ldots + c_k n^k$ and $c_k \neq 0$. Thus, by the equation (1), we obtain $\Delta^k a_n = const \neq 0$. According to Definition 1 we have that the sequence $\{a_n\}$ is the arithmetical sequence of the k-th degree, which completes the proof.

Lemma 1. If sequences $\{a_n^{(j)}\}$, $j = 1, 2, \ldots, l$ satisfy (5), then the sequence $\{u_n\}$ of the form $u_n = A \cdot r_1^{a_n^{(1)}} \cdot r_2^{a_n^{(2)}} \cdot \ldots \cdot r_l^{a_n^{(l)}}$, where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$, satisfies (6).

Proof. Let sequences $\{a_n^{(j)}\}$, for $j = 1, 2, \ldots, l$, satisfy (5). Then, the following condition holds:

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(j)} = 0, \quad j = 1, 2, \ldots, l. \tag{14}$$

So, we have:

$$\prod_{i=0}^{k+1} u_{n+k+1-i}^{(-1)^i \binom{k+1}{i}} = \prod_{i=0}^{k+1} \left(A \cdot r_1^{a_{n+k+1-i}^{(1)}} \cdot \ldots \cdot r_l^{a_{n+k+1-i}^{(l)}} \right)^{(-1)^i \binom{k+1}{i}} =$$

$$= A^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i}} \cdot r_1^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(1)}} \cdot \ldots \cdot r_l^{\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} a_{n+k+1-i}^{(l)}} =$$

$$= A^0 \cdot r_1^0 \cdot \ldots \cdot r_l^0 = 1.$$

It is easy to see that the sequence $\{u_n\}$ satisfies (6).

Corollary 3. If $W_1, W_2, \ldots, W_l$ are polynomials of variable n of degree at most k, then the sequence $\{u_n\}$ of the form: $u_n = A \cdot r_1^{W_1(n)} \cdot \ldots \cdot r_l^{W_l(n)}$, where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$, satisfies (6).

Proof. The sequences $a_n^{(j)} = W_j(n)$ are the arithmetical sequences of at most the k-th degree (Theorem 7), therefore they satisfy the equation (5) (Theorem 4a). Now, by Lemma 1 it follows, that $\{u_n\}$ satisfies (6). Notice, that if $W_1, W_2, \ldots, W_l$ are polynomials of variable n and $d^0(W_j) \leq k$ for $j = 1, 2, \ldots, l$ and at least one of these polynomials is of degree k, then the sequence $u_n = A \cdot r_1^{W_1(n)} \cdot \ldots \cdot r_l^{W_l(n)}$ (where $r_1, r_2, \ldots, r_l \in R_+ \setminus \{1\}$ and $A \neq 0$) is the geometrical sequence of the k-th degree.

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