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SANGAKU FAN SHAPE PROBLEMS

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Abstract

The paper discuss three sangaku problems on relationship among circles inscribed in the sector of an annulus, which is due to its shape, called a fan.

1. Introduction

In the period 17^{th} to 19^{th} century, so called Edo period when Japan closed its doors to the outer world, traditional Japanese mathematics (*wasan*), was developed. In Japan in that times, there was no official academia, so mathematics was developed not only by scholars but also by mathematical laity, that had found mathematics divine. Mathematics enthusiasts dedicated to shrines and temples the wooden tablets on which mathematics problems were written. Those votive tablets are called sangaku. The problems featured on the sangaku are typical problems of japanese mathematics (wasan) and often involve many circles which is uncommon in western mathematics. Each tablet states a theorem or a problem. It is a invitation and a challenge to other experts to prove the theorem or to solve the problem. Most sangaku contain only the final answer to a problem, rarely a detailed solution. It is a work of art as well as a mathematical statement. Sangaku are perishable, and the majority of them have decayed and disappeared during the last two centuries.

2. Main results

The first problem can be found on the top right corner of the Katayamahiko shrine tablet (Fukagawa, Rothman, 2008).

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PROBLEM 1. *As shown in Figure 1, in a sector of annulus of radius R, two circles of radius r are tangent to each other and touch the sector internally. A small circle of radius t touches both the sector and a chord of length d. If $d = 3.62438$ and $2t = 0.34$, find $2r$.*
Tablet contains an answer: $2r = 3,025$.

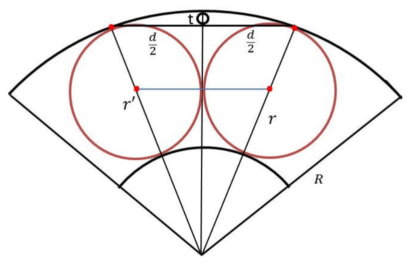

FIGURE 1

Solution. Let O be center of concentric circles that determine the circular ring, C_1 and C_2 two equal circles inscribed in circular ring. Denote by A, B and C the points at which the observed circles touch the edge of the circular ring, with F the touch point of two equal circles, and with D the point where the chord AB touches the small circle (Figure 2).

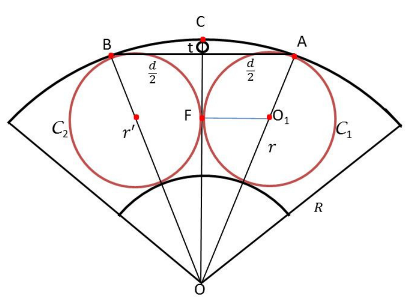

FIGURE 2

Applying the Pythagorean theorem on the rectangular triangle $\triangle ODA$ gives

$$R^2 = (R-2t)^2 + \left(\frac{d}{2}\right)^2$$

which implies

$$R = \frac{d^2}{16t} + t \tag{1}$$

Similarly, using the Pythagorean theorem to $\triangle O_1FO$ (where O_1 is the center of circle C_1), we obtain

$$(R-r)^2 = r^2 + (R-r-2t)^2$$

whence it follow

$$R = r + t + \frac{r^2}{4t} \tag{2}$$

Equating the expressions (1) and (2) for R and solving the resulting quadratic for r gives

$$r = \sqrt{4t^2 + \frac{d^2}{4}} - 2t$$

If $d = 3,62438$ and $2t = 0,34$, then $r = 1,5038$ or $2r = 3,0076$, which is a slightly different result from the one on the Katayamahiko shrine tablet.

The next, central problem in this paper, is given on sangaku in the temple Isaniwa in Ehime Prefecture, well known for its 22 tablets preserved to present day. Tablet (Figure 3) is dated in 1873 (Syomin-no-sanjyutsuten, 2005).

PROBLEM 2. *Let fan makes a third of an annulus, within which one inscribed seven circles: one eastern, two western, two southern and two northern circles. If the diameter of the southern circles given, what is the diameter of the northern circle?*

FIGURE 3

Solution. Denote by C_1 eastern circle, and by C_2, C_3 and C_4 western, southern and northern circle respectively. Let O_1, O_2, O_3, O_4 are centers of circles C_1, C_2, C_3 and C_4 respectively, and r_1, r_2, r_3, r_4 their radii. Let R is the radius of the annulus outer circle and O its center. Let us introduce other symbols as in Figure 4.

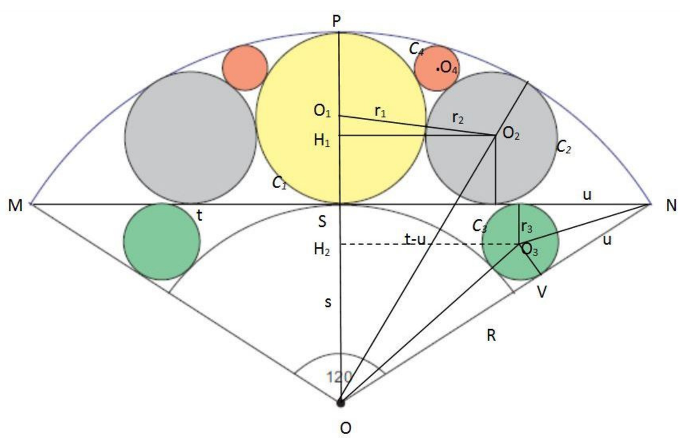

FIGURE 4

In $\triangle OSN$ there are $\angle SON = 60^\circ$ and $\angle OSN = 90^\circ$. Hence,

$$|OS| = s = R \cdot \cos 60^\circ = \frac{R}{2}$$

$$|SN| = t = R \cdot \sin 60^\circ = \frac{R\sqrt{3}}{2}$$

Radius r_1 of the circle C_1 is obtained as follows:

$$|OP| = R \qquad \text{and} \qquad |OP| = |OS| + |SP|$$

i.e.

$$R = s + 2r_1$$

from which we obtain

$$r_1 = \frac{R-s}{2} = \frac{R}{4}.$$

Notice rectangular triangles $\triangle OH_1O_2$ and $\triangle O_1H_1O_2$. In $\triangle OH_1O_2$ there is

$$|H_1O_2|^2 = |OO_2|^2 - |OH_1|^2 \tag{3}$$

and in $\triangle O_1H_1O_2$

$$|H_1O_2|^2 = |O_2O_1|^2 - |O_1H_1|^2. \tag{4}$$

From (3) and (4) we get

$$|OO_2|^2 - |OH_1|^2 = |O_2O_1|^2 - |O_1H_1|^2,$$

respectively

$$(R-r_2)^2 - \left(\frac{R}{2}+r_2\right)^2 = \left(\frac{R}{4}+r_2\right)^2 - \left(\frac{R}{4}-r_2\right)^2.$$

Radius r_2 of the circle C_2 can be expressed using previous equality:

$$r_2 = \frac{3r}{16}. \tag{5}$$

Applying Pythagorean theorem on triangles $\triangle OH_2O_3$ and $\triangle OO_3V$ we obtain

$$|OO_3|^2 = |OH_2|^2 + |H_2O_3|^2$$

and introducing notation $u = |NV|$, previous equality becomes

$$\left(\frac{R}{2}+r_3\right)^2 = \left(\frac{R}{2}-r_3\right)^2 + (t-u)^2.$$

Rearranging the last equality and taking into account (2) we obtain

$$\frac{R\sqrt{3}}{2} - u = \sqrt{2Rr_3}. \tag{6}$$

In $\triangle OO_3V$ there is

$$|OO_3|^2 = |OV|^2 + |VO_3|^2,$$

respectively

$$\left(\frac{R}{2} + r_3\right)^2 = r_3^2 + (R-u)^2 .$$

Finally,

$$\frac{R^2}{4} + Rr_3 = (R-u)^2. \tag{7}$$

Radius r_3 of the circle C_3 and segment $u = |NV|$ can be expressed in terms of R using (6) and (7)

$$r_3 = \frac{3\left(2-\sqrt{3}\right)R}{2\left(2+\sqrt{3}\right)} \cdot u = \frac{3R}{2\left(2+\sqrt{3}\right)}. \tag{8}$$

Lastly, to determine the required radius r_4 of northern circle, we will introduce notations:

$$|O_1H_3| = p,$$

$$|O_4H_3| = q,$$

$$|O_2H_4| = z.$$

In $\triangle OH_3O_4$ there is

$$(R-r_4)^2 = \left(\frac{R}{2} + r_1 + p\right)^2 + q^2$$

and in $\triangle O_1O_2H_1$ there is

$$(r_1+r_2)^2 = (r_1-r_2)^2 + (q+z)^2 ,$$

wheres

$$q + z = 2\sqrt{r_1 r_2}.$$

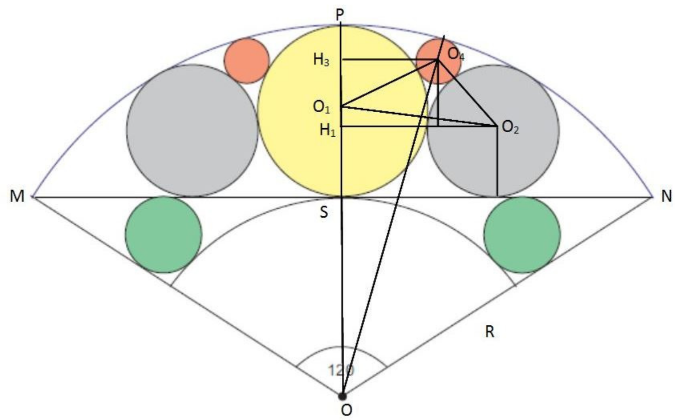

FIGURE 5

Applying the Pythagorean theorem on the rectangular triangles $\triangle O_4H_4O_2$ and $\triangle O_1H_3O_4$ respectively (Figure 5), we obtain

$$(r_3 + r_4)^2 = (p + r_1 - r_2)^2 + z^2,$$
$$p^2 + q^2 = (r_1 + r_4)^2 .$$

Segments p, q, z, r_4 that appeared in previous equalities can be expressed in term if R:

$$r_4 = \frac{3}{193}\left(25 - 12\sqrt{3}\right)R,$$

$$p = \frac{1}{772}\left(-307 + 240\sqrt{3}\right)R,$$

$$q = \frac{2}{193}\left(3 + 14\sqrt{3}\right)R,$$

$$z = \frac{3}{772}\left(-8 + 27\sqrt{3}\right)R.$$

Finally, observing the ratio of radii of circles C_4 and C_3 gives

$$\frac{r_4}{r_3} = \frac{\frac{3}{193}\left(25 - 12\sqrt{3}\right)R}{\frac{3\left(2-\sqrt{3}\right)R}{2\left(2+\sqrt{3}\right)}} = \cdots = \frac{62 + \sqrt{3 \cdot 1024}}{193}$$

and

$$r_4 = r_3 \cdot \frac{62 + \sqrt{3 \cdot 1024}}{193}. \tag{9}$$

Equality (9) corresponds to the solution stated on sangaku in temple Isaniwa.

The third problem dates back to 1865. and is given on sangaku in Meiseirinji temple (Syomin-no-sanjyutsuten, 2005). In solution of this problem inversion technique will be used. Theorem of inversion of circles will be stated without proof.

Theorem 1. *A circle, its inverse, and the center of inversion are collinear.*

PROBLEM 3. *Inside a fan-shaped sector five circles touch each other; one is a "red" circle of radius r_1, two are "green" circles of radius r_2, and two are "white" circles of radius r_3. The radius of the sector is r, and the circles touch each other symmetrically about the center O. We take the angle of the sector to be variable and r constant. As the angle is varied, the inner radius of the sector t is adjusted so that the five circles continue to touch; r_3 is also allowed to vary, while the other radii remain constant.*
Show that $2\left(r_1 + r_3\right) = r$, when r_3 is a maximum.

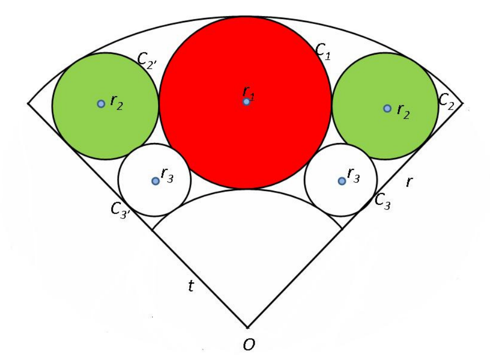

FIGURE 6

Solution. Let C_1 denote "red" circle, C_2 and $C_{2'}$ two "green" circles and C_3 and $C_{3'}$ two "white" circles, as in Figure 6. Under conditions of the problem, it is sufficient to consider one half of the figure given.
Assume initially that the center O and the centers O_2 and O_3 of the circles C_2 and C_3 are collinear. Then, Figure 7 shows that

$$r = t + 2r_1 = t + 2r_3 + 2r_2 \tag{10}$$

or

$$r_1 = r_2 + r_3. \tag{11}$$

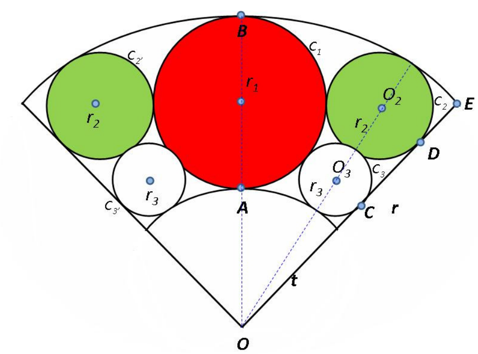

FIGURE 7

Similar triangles $\triangle OO_3C$ and $\triangle OO_2D$ give

$$\frac{r_3}{t + r_3} = \frac{r_2}{t + 2r_3 + r_2}.$$

Eliminating t by previous equality and using (10) give

$$\frac{r_3}{r - 2r_2 - r_3} = \frac{r_2}{r - r_2}$$

or

$$r_3 = \frac{1}{r}\left(-2r_2^2 + rr_2\right). \tag{12}$$

Expression (12) can be rewritten as

$$r_3 = \frac{1}{r}\left[-2\left(r_2 - \frac{r}{4}\right)^2 + \frac{r^2}{8}\right].$$

Given condition of constant radius r, last expression implies that r_3 is maximized and equals $\frac{r}{8}$ when $r_2 = \frac{r}{4}$. This and (11) imply

$$2r_1 + 2r_3 = 2r_2 + 4r_3 = r.$$

Therefore, the statement is proven in case of collinear centers of circles with radii r, r_2 and r_3.
It remains to prove that the aforementioned centers of circles collinear. In this purpose, consider a Figure 7 and make use of Theorem 1. Choosing O as the center of inversion, if we can invert r_2 into r_3 and vice versa, we have shown that the two circles are collinear with O, and the rest of the proof follows.
To do this, notice that if in Figure 7 we invert circle with radius t into circle with radius r, and vice versa, then circle with radius r_1 must invert into itself in order to keep the points of tangency A and B invariant. Similarly, circles with radii r_3 and r_2 are tangent to circle with radius r_1 and to the line OE at the points C and D. In order that all points of tangency are preserved, in particular that C inverts into D and vice versa, then circle with radius r_2 must invert into circle with radius r_3, and the reverse. To do this, merely choose the radius of inversion k such that $k^2 = rt$.

3. Final remarks

In the Edo era of the 18^{th} and 19^{th} centuries in Japan, ordinary people enjoyed mathematics in daily life, not as a professional study but rather as an intellectual popular game and a recreational activity. Sangaku usually don't provide a proof of the theorem, and even books of them have been published in Japan for many years, some theorems are still unsolved. It gives opportunity to researchers to explore and decrypt sangaku problems as well as to link similar problems. Sangaku can be used to stimulate the interest of students in mathematics as many of sangaku problems are a source of pleasure and challenge.

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