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SOME REMARKS ON STRONG SEQUENCES

JOANNA JURECZKO

Abstract

Strong sequences were introduced by Efimov in the 60s' of the last century as a useful method for proving well known theorems on dyadic spaces i.e. continuous images of the Cantor cube. The aim of this paper is to show relations between the cardinal invariant associated with strong sequences and well known invariants of the continuum.

1. Introduction

Strong sequences were introduced by B. A. Efimov in [4], as a useful tool for proving well known theorems on dyadic spaces. Among others he proved that strong sequences do not exist in the subbase of the Cantor cube. This is our opinion that it could be interesting the answer of the natural question about properties of spaces in which strong sequences exist and consequences of such existence. This is how the interest of the strong sequences method was born, (for further historical notes concerning strong sequences see [6]). Particularly, strong sequences method, as was shown in e.g. [7, 8] is equivalent to partition theorems. Moreover, if we associate the cardinal invariant with the length of strong sequences in spaces where such sequences exist, we can obtain interesting results, (see also [8, 9]). This is our hope that this invariant can be usefull characterisation of such spaces.

In this paper we will consider the space $(\omega^\omega, \leq^*)$ in which, as we will show, strong sequences exist. We will investigate inequalities between invariant $\hat{\mathfrak{s}}$ associated with strong sequences and other well known invariants like: boundeness, covering number and the invariant associated with MAD families.

Our paper is organized as follows. In section 2 we gather all definitions and previous facts needed for further parts of this paper. In Section 3 we show main results. The paper is finished by some results in forcing,

• *Joanna Jureczko* — e-mail: joanna.jureczko@pwr.edu.pl
Wrocław University of Science and Technology.

(Section 4) in which we will show some strong inequalities which can be obtained between $\hat{\mathfrak{s}}$ and considered invariants. In this part we give some open problems.

2. Definitions and previous results

1. Consider a *partially preordered* set $(X, \preceq)$, i.e. a set ordered by reflexive and transitive relation $\preceq$. Let $a, b, c, x \in X$. We say that a and b are *comparable* iff $a \preceq b$ or $b \preceq a$. We say that a and b are *compatible* iff there exists $c \in X$ such that $a \preceq c$ and $b \preceq c$. (In this case we say that a, b have a *bound*). A set $A \subset X$ is called an *ω-directed set* iff every subset of A of cardinality less than ω has a bound which belongs to A.

Definition 1. *A sequence $(S_\phi, H_\phi)_{\phi<\alpha}$, where $S_\phi, H_\phi \subset X$, and $|S_\phi| < \omega$ is called a strong sequence if:*
1^o *$S_\phi \cup H_\phi$ is ω-directed for all $\phi < \alpha$;*
2^o *$S_\psi \cup H_\phi$ is not ω-directed, for all ψ and ϕ such that $\phi < \psi < \alpha$.*

In [6] the strong sequence number $\hat{\mathfrak{s}}(X)$ was introduced as follows:

(1) $$\hat{\mathfrak{s}}(X) = \sup\{\kappa\colon \text{there exists a strong sequence on } X \text{ of length } \kappa\}.$$

2. We say that $(X, \preceq)$ iff $\preceq$ is reflexive and transitive.
A subset $B \subset X$ is called *bounded* iff B has a bound. The set which is not bounded will be called *unbounded.*
A subset $A \subseteq B \subseteq X$ is called *cofinal* in B iff for any $b \in B$ there exists $a \in A$ such that $b \preceq a$. A cofinal subset in the whole set X is called also *a dominating set.* The following invariants are well known:

(2) $$\mathfrak{b}(X) = \min\{|A|\colon A \subset X \wedge A \text{ is unbounded in } X\},$$

(3) $$\mathfrak{d}(X) = \min\{|A|\colon A \subset X \wedge A \text{ is cofinal in } X\}.$$

Fact 1 ([3]). *Let $(X, \preceq)$ be a partially preordered set without maximal elements. Then $\mathfrak{b}(X)$ is regular and*

(4) $$\mathfrak{b}(X) \leq cf(\mathfrak{d}(X)) \leq \mathfrak{d}(X).$$

3. We will provide our considerations for $(X, \preceq) = (\omega^\omega, \leq^*)$, i.e. in the set of all functions $\omega \to \omega$ ordered by

(5) $$f \leq^* g \text{ iff } |\{n \in \omega\colon g(n) < f(n)\}| < \omega.$$

We accept the notation: $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}(\omega^\omega), \mathfrak{b} = \mathfrak{b}(\omega^\omega), \mathfrak{d} = \mathfrak{d}(\omega^\omega)$.

4. A family $\mathcal{I}$ of subsets of X which satisfies the following three conditions

1) $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$;
2) $\{x\} \in \mathcal{I}$ for all $x \in \mathcal{I}$;
3) $X \notin \mathcal{I}$
is called *a family of thin sets.*
A subfamily $\mathcal{B} \subset \mathcal{I}$ is called *a base* of the family $\mathcal{I}$ of thin sets iff for each set $A \in \mathcal{I}$ there exists a set $B \in \mathcal{B}$ such that $A \subseteq B$.

We remind definitions of the following invariants, (see e. g. [3] p.250):

$$\operatorname{add}(\mathcal{I}) = \min\left\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\right\} \tag{6}$$

$$\operatorname{cov}(\mathcal{I}) = \min\left\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \bigcup \mathcal{A} = X\right\} \tag{7}$$

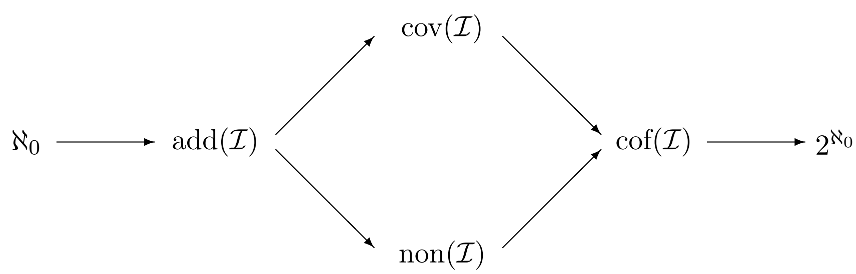
$$\operatorname{non}(\mathcal{I}) = \min\{|A| \colon A \notin \mathcal{I} \wedge A \in \mathcal{P}(X)\} \tag{8}$$

$$\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}| \colon \mathcal{A} \subset \mathcal{I} \wedge \mathcal{A} \text{ is a base of } \mathcal{I}\}. \tag{9}$$

Notice that any ideal on X is a family of thin sets. (Clearly, $\mathcal{I}$ is an ideal iff $\operatorname{add}(\mathcal{I}) \geq \aleph_0$).

The following diagram is known in the literature as "Cichoń diagram" and was introduced by Fremlin in [5]. Since that paper the diagram has been completed and modified by many authors. Below we remind this diagram for four invariants defined above.

Fact 2 ([1])**.** *If $\mathcal{I}$ is a family of thin sets, then*



where $\alpha \to \beta$ denotes $\alpha \leq \beta$.

5. Let $\mathbb{R}$ be the real line with standard topology. Let μ be the Lebesque measure on $\mathbb{R}$. Then

$$\mathcal{M} = \{A \subset \mathbb{R} \colon A \text{ is meager}\}, \tag{10}$$

$$\mathcal{N} = \{A \subset \mathbb{R} \colon \mu(A) = \emptyset\}. \tag{11}$$

Notice, that $\mathcal{M}$ and $\mathcal{N}$ are both ideals.

6. In [1] one can find the following results:

Fact 3 (Bartoszyński) *$cov(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}}|\{n\in\omega\colon f(n)\neq g(n)\}|<\omega. \tag{12}$$

Fact 4 (Keremedis) *$non(\mathcal{M})$ is the cardinality of the smallest family $\mathcal{F} \subseteq \omega^\omega$ such that*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}}|\{n\in\omega\colon f(n)= g(n)\}|<\omega. \tag{13}$$

Fact 5 (Rothberger)

$$\mathrm{cov}(\mathcal{M})\leq \mathrm{non}(\mathcal{N}) \text{ and } \mathrm{cov}(\mathcal{N})\leq \mathrm{non}(\mathcal{M}). \tag{14}$$

Fact 6 (Bartoszyński, Raisonnier and Stern)

$$\mathrm{add}(\mathcal{N})\leq \mathrm{add}(\mathcal{M}), \tag{15}$$

$$\mathrm{cof}(\mathcal{M})\leq \mathrm{cof}(\mathcal{N}). \tag{16}$$

Fact 7 (Miller, Truss)

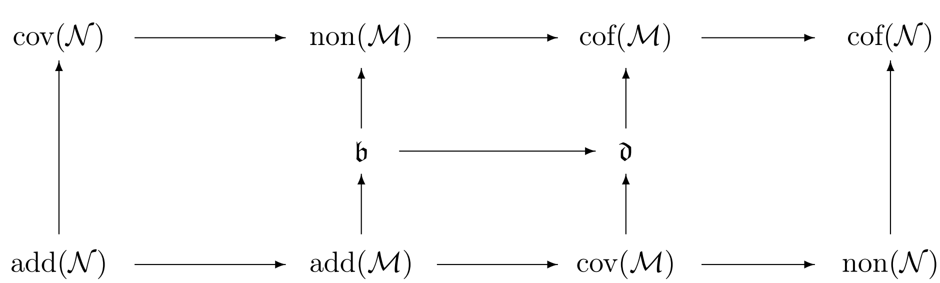
$$\mathrm{add}(\mathcal{M})=\min\{\mathrm{cov}(\mathcal{M}),\mathfrak{b}\}. \tag{17}$$

Fact 8 (Fremlin)

$$\mathrm{cof}(\mathcal{M})=\max\{\mathrm{non}(\mathcal{M}),\mathfrak{d}\}. \tag{18}$$

According to equalities (14) - (18) the following diagram holds:

Fact 9 ([1]).



where $\alpha \to \beta$ denotes $\alpha \leq \beta$.

Observation 1. ***(i)*** *Let $\mathcal{F} \subseteq \omega^\omega$ be the smallest family of the property*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}} |\{n \in \omega\colon f(n) = g(n)\}| < \omega.$$

Then $|\{n \in \omega\colon f_\alpha(n) \neq f_\beta(n)\}| = \omega$ for all $f_\alpha, f_\beta \in \mathcal{F}, \alpha \neq \beta$.
(ii) *Let $\mathcal{F} \subseteq \omega^\omega$ be the smallest family of the property*

$$\forall_{g\in\omega^\omega}\exists_{f\in\mathcal{F}} |\{n \in \omega\colon f(n) \neq g(n)\}| < \omega.$$

Then $|\{n \in \omega\colon f_\alpha(n) \neq f_\beta(n)\}| = \omega$ for all $f_\alpha, f_\beta \in \mathcal{F}, \alpha = \beta$.

Proof. We prove (i) only, (ii) can be proved similarly but using Fact 3.
(i) By Fact 4 we have $|\mathcal{F}| = \mathrm{non}(\mathcal{M})$. Suppose in contrary that there are $\alpha \neq \beta$ such that $|\{n \in \omega\colon f_\alpha(n) = f_\beta(n)\}| = \omega$. Let

$$A(\alpha, \beta) = \{n \in \omega\colon f_\alpha(n) = f_\beta(n)\}.$$

Let $\{g_\gamma \in \omega^\omega \setminus \mathcal{F}\colon \gamma < \eta\}$ be a family such that

$$|\{n \in \omega\colon g_\gamma(n) = f_\beta(n)\}| < \omega$$

for all $\gamma < \eta$. Let $B(\gamma, \beta) = \{n \in \omega\colon g_\gamma(n) = f_\beta(n)\}$ for all $\gamma < \eta$. Obviously $|A(\alpha, \beta) \cap B(\gamma, \beta)| < \omega$. Then $g_\gamma(n) = f_\alpha(n)$ for all $n \in A(\alpha, \beta) \cap B(\gamma, \beta)$. A contradiction with the minimality of $\mathcal{F}$. □

7. Two functions $f, g \in \omega^\omega$ are *almost disjoint* iff there are finite values of $\alpha \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ such that $f(\alpha) = g(\alpha)$. When the functions have domain ω almost disjointness means that they are eventually different ($f(\alpha) \neq g(\alpha)$) for all sufficiently large $\alpha < \omega$. A *maximal almost disjoint (MAD) family* of functions on ω is an almost disjoint family of functions $\omega \to \omega$ that is not properly included in another such family. In [2] the following invariant is associated with MAD families of functions:

(19) $$\mathfrak{a}_e = \min\{\mathcal{A} \subseteq P(\omega^\omega) : \mathcal{A} \text{ is a MAD family}\}.$$

Fact 10 ([2]).

(20) $$\mathfrak{a}_e \geq \omega^+.$$

Observation 2.

(21) $$\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_e.$$

Proof. Immediately by Fact 4. □

3. Main results

Theorem 1.

(22) $$\mathfrak{b} \leq \hat{\mathrm{s}}.$$

Proof. Suppose that $\hat{\mathrm{s}} < \mathfrak{b}$ and $\kappa \leq \hat{\mathrm{s}}$. Let $\{(S_\alpha, H_\alpha) : \alpha < \kappa\}$ be a maximal strong sequence in ω^ω. For any $\alpha < \kappa$ define

$$A_\alpha = \{f \in S_\alpha \setminus H_\beta \colon \{f\} \cup H_\beta \text{ is not } \omega\text{-directed for } \beta < \alpha\}.$$

Define an increasing function

$$F \colon \kappa \to \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha).$$

such that

$$F(\alpha) = \begin{cases} f_\alpha \in H_\alpha & \text{for } \alpha = 0; \\ f_\alpha \in A_\alpha & \text{for } \alpha > 0. \end{cases}$$

Since ω^ω has no maximal elements, this function is well-defined.

Let

$$A = \{f_\alpha \in A_\alpha \colon f_\alpha = F(\alpha), \alpha < \kappa\}.$$

Since $\kappa < \mathfrak{b}$, there exists $g \in A$ such that $f_\alpha \leq g$, for all $f_\alpha \in S_\alpha$. As ω^ω has no maximal elements, there exists $h \in \omega^\omega \setminus \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha)$ such that $g < h$. Thus there exists a maximal ω-directed set $S \subset \omega^\omega \setminus \bigcup_{\alpha<\kappa} (S_\alpha \cup H_\alpha)$ such that $h \in S$ and $S \cup H_\alpha$ is not ω-directed for any $\alpha < \kappa$. A contradiction with maximality of the strong sequence $\{(S_\alpha, H_\alpha) : \alpha < \kappa\}$. □

Theorem 2.

(23) $$cov(\mathcal{M}) \leq \hat{s}.$$

Proof. Let $\mathrm{cov}(\mathcal{M}) = \kappa$. By Fact 3 there exists the smallest family

$$\mathcal{F} = \{f_\alpha \in \omega^\omega \colon \alpha < \kappa\}$$

fulfilling (12)

Thus we can construct a function $H \colon \omega^\omega \to \kappa$ such that

$$H(g) = \min\{\alpha \colon |\{n \in \omega \colon f_\alpha(n) = g(n)\}| = \omega\}.$$

The family $\mathcal{F}$ is well-ordered hence the function H is well-defined.
We will construct a strong sequence in ω^ω with relation defined as follows:

$$\text{if } f_\alpha \in \mathcal{F}, \text{ then } f_\alpha \preceq g \text{ iff } h(g) = \alpha;$$

$$\text{if } f \notin \mathcal{F}, \text{ then } f \preceq g \text{ iff } |\{n \in \omega \colon f(n) = g(n)\}| = \omega.$$

Let $g_0 \in \omega^\omega$ be an arbitrary function. Then there exists $f \in \mathcal{F}$ such that $|\{n \in \omega \colon f(n) = g_0(n)\}| = \omega$. Let $f_{\alpha_0} \in \mathcal{F}$ be a function such that $h(g_0) = \alpha_0$. Let $\mathcal{S}_0 = \{g_0\}$ and $\mathcal{H}_0 = \{g \in \omega^\omega \colon h(g) = \alpha_0\}$. Obviously $\mathcal{H}_0$ is non-empty. Let $(\mathcal{S}_0, \mathcal{H}_0)$ be the first element of a strong sequence.

Since $\mathcal{H}_0 \neq \omega^\omega$ there exists $g_1 \in \omega^\omega \setminus \mathcal{H}_0$ such that $h(g_1) \neq \alpha_0$. Hence we can construct the next element of the strong sequence. Let $f_{\alpha_1} \in \mathcal{F}$ be a fucntion such that $|\{n \in \omega\colon g_1(n) = f_{\alpha_1}(n)\}| = \omega$. Let $\mathcal{S}_1 = \{g_1\}$ and $\mathcal{H}_1 = \{g \in \omega^\omega \setminus \mathcal{H}_0\colon h(g) = \alpha_1\}$.

Assume that the strong sequence $\{(\mathcal{S}_\gamma, \mathcal{H}_\gamma) : \gamma < \beta\}$ such that

$$(\mathcal{S}_\gamma, \mathcal{H}_\gamma) = \left(\{g_\gamma\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\delta\colon \delta < \gamma\} : h(g) = \alpha_\gamma\right\}\right),$$

where $g_\gamma \in \omega^\omega \setminus \bigcup_{\delta<\gamma} H_\delta$, has been defined,.

Since $\beta < \kappa$ and by Observation 1, there exists $g_\beta \in \omega^\omega \setminus \left\{f_{\alpha_\gamma}\colon \gamma < \beta\right\}$ be a function such that $|\left\{n \in \omega\colon g_\beta(n) = f_{\alpha_\beta}(n)\right\}| = \omega$. Let

$$(\mathcal{S}_\beta, \mathcal{H}_\beta) = \left(\{g_\beta\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\gamma\colon \gamma < \beta\right\} : h(g) = \alpha_\beta\}\right).$$

Thus the strong sequence of length $|\mathcal{F}|$ has been constructed. □

Theorem 3.

$$\mathfrak{a}_e \leq \hat{s}. \tag{24}$$

Proof. By Fact 8 we have $\mathfrak{a}_e \geq \omega^+$. Let $\mathcal{F}_e$ be a MAD family of functions $\omega \to \omega$ of cardinality ω^+. We will construct a strong sequence of cardinality ω^+ in ω^ω with the following relatio:

$$f \preceq g \text{ iff } |\{\alpha \in \omega\colon f(\alpha) = g(\alpha)\}| = \omega.$$

Let $f_0 \in \mathcal{F}_e$ be a function. Let $(\mathcal{S}_0, \mathcal{H}_0) = (\{f_0\}, \{g \in \omega^\omega\colon f_0 \preceq g\})$ be the first element of a strong sequence. Obviously $(\mathcal{S}_0, \mathcal{H}_0)$ is non-empty because $f_0 \in \mathcal{H}_0$. Let $f_1 \in \mathcal{F}_e \setminus \mathcal{H}_0$. Let $(\mathcal{S}_1, \mathcal{H}_1) = (\{f_1\}, \{g \in \omega^\omega\colon f_1 \preceq g\})$. By our construction $\mathcal{H}_0 \cup \mathcal{H}_1$ is not ω-directed. Let $(\mathcal{S}_1, \mathcal{H}_1)$ be the second element of the strong sequence.

Assume that the strong sequence $\{(\mathcal{S}_\gamma, \mathcal{H}_\gamma) : \gamma < \beta < \omega^+\}$ such that

$$(\mathcal{S}_\gamma, \mathcal{H}_\gamma) = \left(\{f_\gamma\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\delta\colon \delta < \gamma\colon f_\gamma \preceq g\right\}\right),$$

where $f_\gamma \in \mathcal{F}_e \setminus \bigcup\{\mathcal{H}_\delta\colon \delta < \gamma\}$, has been defined.

Since $\beta < \omega^+$ there exists $f_\beta \in \mathcal{F}_e \setminus \bigcup\{\mathcal{H}_\gamma\colon \gamma < \beta\}$. Let

$$(\mathcal{S}_\beta, \mathcal{H}_\beta) = \left(\{f_\beta\}, \left\{g \in \omega^\omega \setminus \bigcup\{\mathcal{H}_\delta\colon \gamma < \beta\colon f_\beta \preceq g\right\}\right),$$

Thus the strong sequence of length $|\mathcal{F}|$ has been constructed. □

Corollary 1.

$$non(\mathcal{M}) \leq \mathfrak{a}_e \leq \hat{s}. \tag{25}$$

Proof. Immediately by Fact 10 and Theorem 3. □

Theorem 4. *In $(\omega^\omega, \leq^*)$ there exists a strong sequence of length $2^{\aleph_0}$.*

Proof. Fix a MAD family of sets $\mathcal{A} = \left\{A_\alpha \subseteq [\omega]^\omega : \alpha < 2^{\aleph_0}\right\}$, (i.e. a family of infinite subsets of ω such that $|A \cap B| < \omega$ for any $A, B \in \mathcal{A}$). For each $A \in \mathcal{A}$ consider functions: $F_n^A \in \omega^\omega$ such that

$$F_n^A(a) = \begin{cases} n+1 & \text{for } a \in A \\ 0 & \text{for } a \notin A \end{cases}$$

and $F_\omega^A \in \omega^\omega$ such that

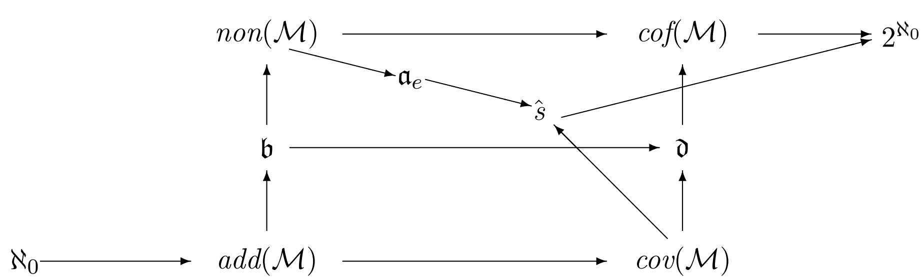
$$F_\omega^A(a) = \begin{cases} a & \text{for } a \in A \\ 0 & \text{for } a \notin A. \end{cases}$$

Obviously

$$F_0^A <^* F_1^A <^* \dots <^* F_\omega^A.$$

Now take $(S_A, H_A) = \left(\{F_\omega^A\}, \left\{F_n^A : n < \omega\right\}\right)$. Then $S_A \cup H_A$ is ω-directed, because F_ω^A is its bound. Now take $A_\alpha, A_\beta \in \mathcal{A}$ such that $\alpha < \beta$. Then $S_{A_\beta} \cup H_{A_\alpha}$ is not ω-directed, because it contains no bound for H_{A_α}. Since all MAD families have cardinality $2^{\aleph_0}$ we obtain that $\{(S_A, H_A) : A \in \mathcal{A}\}$ is the required strong sequence. □

Corollary 2. *The following diagram holds*



where $\alpha \to \beta$ means $\alpha \leq \beta$:

Proof. Immediately by equalities (4), (17), (18) and Theorems 1-4. □

4. Some results for forcing notion

According to [1] pp. 380-397, the following inequalities are consistent with ZFC.

In the iterated Cohen's model with finite supports $\mathrm{non}(\mathcal{M}) = \aleph_1 \wedge \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ which is connecting with Cichoń diagram we have $\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \mathfrak{b} = \aleph_1$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{r} = \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathfrak{c} > \aleph_1$. Thus

$$\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{b} < \hat{\mathrm{s}}. \tag{26}$$

By adding $\aleph_2$ random reals a model of CH we have $\mathrm{non}(\mathcal{N}) = \aleph_1 < \mathrm{cov}(\mathcal{N}) = \aleph_2 = \mathfrak{c}$. Thus

$$\mathrm{non}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{27}$$

By adding $\aleph_2$ Hechler's reals (with finite support) to a model of CH we get $\mathrm{cov}(\mathcal{N}) = \aleph_1 < \mathrm{add}(\mathcal{M}) = \aleph_2 = \mathfrak{c}$. Hence it is consistent that

$$\mathrm{cov}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{28}$$

Alternatively adding $\aleph_2$ Cohen and Laver reals (with countable support) over a model of CH we have $\mathrm{cov}(\mathcal{N}) = \aleph_1 < \mathrm{add}(\mathcal{M}) = \aleph_2 = \mathfrak{c}$ Thus

$$\mathrm{cov}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{29}$$

Alternatively iterating $\aleph_2$ times rational perfect forcing and Roslanowski-Shelah forcing over a model of CH we obtain $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) = \mathfrak{d} = \aleph_2$. Therefore, it is consistent that

$$\mathrm{cov}(\mathcal{M}) < \hat{\mathfrak{s}}. \tag{30}$$

Finally in the iterated Sachs model we have that $\mathrm{cof}(\mathcal{N}) = \aleph_1$. Hence, it is consistent with ZFC that

$$\mathrm{cof}(\mathcal{N}) < \hat{\mathfrak{s}}. \tag{31}$$

Open problem. Is there any relation between
a) $\hat{\mathfrak{s}}$ and $\mathrm{cof}(\mathcal{M})$?
b) $\hat{\mathfrak{s}}$ and $\mathrm{non}(\mathcal{N})$?
c) $\hat{\mathfrak{s}}$ and $\mathrm{cof}(\mathcal{N})$?
d) $\hat{\mathfrak{s}}$ and $\mathfrak{d}$?

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Joanna Jureczko
Wrocław University of Science and Technology,
Faculty of Electronics
Janiszewskiego Street 11/17, 50-372 Wrocław
Email address: joanna.jureczko@pwr.edu.pl