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THE USE OF FINITE EXPANSION OF FUNCTIONS FOR EVALUATION OF LIMITS

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The theory of finite expansions of functions is very helpful in evaluation of complicated limits. One-variable functions are replaced by appropriate polynomials. Extensive chapters in French textbook are devoted to the theory of finite expansions and its applications. In Polish mathematical literature the problem of finite expansions is omitted. More complicated limits are evaluated using l'Hôpital's rule or Taylor and Maclaurin series. However, there exists close connection between those series and finite expansions. The goal of this paper is popularization of the theory of finite expansions on the Polish ground.

The symbol $(DL)_n^{x_o}(f)$ will be used for finite expansion of the nth degree of a function f in the neighborhood of point x_0, whereas the symbol $d^o(P)$ means the degree of one-variable polynomial P with real coefficients.*

Definition 1.

a)

$$(DL)_n^0(f) = P \Leftrightarrow \left\{ d^o(P) \leqslant n \wedge \lim_{x\to 0} \frac{1}{x^n}\left[f(x) - P(x)\right] = 0 \right\};$$

b)

$$g(x) = f(x + x_0) \Rightarrow (DL)_n^{x_0}(f) = (DL)_n^0(g);$$

*The symbol DL is an abbreviation of "developpement limite" (finite expansion).

c)

If a function f is defined in the interval $(x_0, +\infty)$ and $g(x) = f\left(\frac{1}{x}\right)$, then $(DL)_n^{+\infty}(f) = (DL)_n^0(g)$.

According to the above definition the finite expansion of a function f in the neighborhood of the point x_0 is a certain polynomial.

Example 1.
A polynomial $P(x) = 1+x+x^2+...+x^n$ is the finite expansion of the nth degree of the function $f(x) = \frac{1}{1-x}$ in the neighborhood of zero, i.e. $(DL)_n^0(f) = P$.

Actually, $d^o(P) = n$, $\lim\limits_{x\to 0} \frac{1}{x^n}[f(x) - P(x)] = \lim\limits_{x\to 0} \frac{1}{x^n}[\frac{1}{1-x} - (1+x+x^2+...+x^n)] =$

$$= \lim_{x\to 0} \frac{1}{x^n(1-x)}[1-(1-x)(1+x+x^2+...+x^n)] =$$

$$= \lim_{x\to 0} \frac{1}{x^n(1-x)} \cdot [1-(1-x^{n+1})] = \lim_{x\to 0} \frac{x^{n+1}}{x^n(1-x)} = \lim_{x\to 0} \frac{x}{1-x} = 0.$$

Theorem 1.
If a function f is defined on an open interval I and $0 \in I$, $n \geqslant 0$, then this function has at most one finite expansion of the degree n in the neighborhood of the point $x = 0$.

Proof.
Suppose that $(DL)_n^0(f) = P$ and $(DL)_n^0(f) = Q$. Then according to Definition (1a) we have:

$$d^o(P) \leqslant n \wedge d^o(Q) \leqslant n. \tag{1}$$

$$\lim_{x\to 0} \frac{1}{x^n}[f(x) - P(x)] = 0 \wedge \lim_{x\to 0} \frac{1}{x^n}[f(x) - Q(x)] = 0. \tag{2}$$

Consider a polynomial R of the form:

$$R(x) = P(x) - Q(x) = C_0 + C_1 x + ... + C_k x^k, \qquad \text{where } x \in I,\ k \leqslant n.$$

Suppose also by contradiction that $R(x) \neq 0$. Then at least one of coefficients of this polynomial is non-zero, for example, $C_{j_1} \neq 0$ $(j_1 \leq k)$.
Let $l = \inf\{j : C_j \neq 0\}$.

Then we have:

$$\frac{R(x)}{x^n} = \frac{1}{x^n}(0+0\cdot x+...+0\cdot x^{l-1}+C_l x^l+...+C_k x^k) = \frac{1}{x^{n-l}}(C_l+C_{l+1}x+...+C_k x^{k-l}).$$

If $l = n$, then:

$$\lim_{x\to 0} \frac{R(x)}{x^n} = \lim_{x\to 0} \frac{1}{x^{n-l}} \left(C_l + C_{l+1}x + ... + C_k x^{k-l}\right) = C_l \neq 0. \tag{3}$$

If $l < n$, then:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = +\infty. \tag{4}$$

However, we have:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = \lim_{x\to 0} \left|\frac{P(x)-Q(x)}{x^n}\right| = \lim_{x\to 0} \left|\frac{f(x)-Q(x)}{x^n} - \frac{f(x)-P(x)}{x^n}\right| \leqslant$$

$$\leqslant \lim_{x\to 0} \left(\left|\frac{f(x)-Q(x)}{x^n}\right| + \left|\frac{f(x)-P(x)}{x^n}\right|\right),$$

from which on the basis of (2) we obtain:

$$\lim_{x\to 0} \left|\frac{R(x)}{x^n}\right| = 0 \qquad \text{and therefore} \qquad \lim_{x\to 0} \frac{R(x)}{x^n} = 0,$$

what contradicts (3) and (4). Hence, assumption that $P(x) - Q(x) \neq 0$ leads to contradiction, so $P(x) - Q(x) = 0$ and $P = Q$.

Definition 2.
If

$$P(x) = a_0 + a_1 x + ... + a_n x^n, 0 \leqslant m \leqslant n \quad \text{and} \quad \varphi_m : R^{n+1}[x] \to R^{n+1}[x]$$

$$\text{and} \quad \varphi_m(P(x)) = a_0 + a_1 x + ... + a_m x^m,$$

then the polynomial $\varphi_m(P(x))$ is called the mth degree restriction of a polynomial $P(x)$.*

Let us prove several properties of finite expansions which will be used for evaluation of limits.

Theorem 2.
If a function f is defined on an open interval I and $0 \in I$, $d^o(P) \leqslant n$, then:

$$DL_n^0(f) = P \Leftrightarrow \underset{\varepsilon : I \to R}{\exists} [\lim_{h\to 0} \varepsilon(h) = 0 \wedge \underset{x\in I}{\forall} f(x) = P(x) + x^n \varepsilon(x)].$$

*$R^{m+1}[x]$ is a set of all polynomials in one variable x of at most nth degree.

To prove it is enough to suppose that

$$\varepsilon(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{x^n}[f(x) - P(x)], & \text{if } x \neq 0. \end{cases}$$

At the end of this paper we cite Table 1 with finite expansions of important functions in the neighborhood of the point $x = 0$, i.e. having the form of $f(x) = P(x) + x^n \varepsilon(x)$, where $\varepsilon(x) \underset{x \to 0}{\to} 0$.

Theorem 3.

$$(DL)_n^0(f) = P \wedge 0 \leqslant m \leqslant n \Rightarrow (DL)_m^0(f) = \varphi_m(P).$$

Proof.
From the assumption and Theorem 2 it follows that $f(x) = P(x) + x^n \varepsilon_1(x)$, where $\varepsilon_1(x) \underset{x \to 0}{\to} 0$. Moreover, $d^o(\varphi_m(P)) = m$. A function f can be written as

$$f(x) = \varphi_m(P(x)) + x^{m+1} \cdot S(x) + x^n \varepsilon_1(x) = \varphi_m(P(x)) + x^m [xS(x) + x^{n-m} \varepsilon_1(x)],$$

with $\lim\limits_{x \to 0} [xS(x) + x^{n-m} \varepsilon_1(x)] = 0$.

On the basis of Theorem 2 we assert that $(DL)_m^0(f) = \varphi_m(P)$.

Theorem 4.

$$\text{If } (DL)_n^0(f) = P \text{ and } (DL)_n^0(g) = Q, \text{ then}$$

a) $(DL)_n^0(\lambda_1 f + \lambda_2 g) = \lambda_1 P + \lambda_2 Q$
(theorem about finite expansion of linear combination of two functions);

b) $(DL)_n^0(f \cdot g) = \varphi_n(P \cdot Q)$
(theorem about finite expansion of a product of two functions);

c) if $g(0) \neq 0$ and χ is a quotient of the nth degree from division of a polynomial P by a polynomial Q according to growing powers (i.e. $P(x) = Q(x) \cdot \chi(x) + x^{n+1} Q_n(x)$ i $d^o(\chi) = n$), then $(DL)_n^0(\frac{f}{g}) = \chi$
(theorem about finite expansion of the division of two functions).

Proof.
It follows from the assumption that

$$d^o(P) \leqslant n, \; d^o(Q) \leqslant n, \tag{5}$$

$$\lim_{x \to 0} \frac{1}{x^n} [f(x) - P(x)] = 0, \; \lim_{x \to 0} \frac{1}{x^n} [g(x) - Q(x)] = 0. \tag{6}$$

a) From (5) and (6) we have:

$$d^o(\lambda_1 P + \lambda_2 Q) \leqslant n,$$

$$\lim_{x\to 0}\frac{1}{x^n}[\lambda_1 f(x) + \lambda_2 g(x) - (\lambda_1 P(x) + \lambda_2 Q(x))] =$$

$$= \lim_{x\to 0}\frac{1}{x^n}[\lambda_1(f(x) - P(x)) + \lambda_2(g(x) - Q(x))] =$$

$$= \lambda_1 \lim_{x\to 0}\frac{1}{x^n}[(f(x) - P(x)] + \lambda_2 \lim_{x\to 0}\frac{1}{x^n}[(g(x) - Q(x))] = 0$$

Hence, on the basis of Definition (1a) we assert that

$$(DL)_n^0(\lambda_1 f + \lambda_2 g) = \lambda_1 P + \lambda_2 Q.$$

b) It follows from the assumptions of theorem that $d^o(\varphi_n(P\cdot Q)) \leqslant n$ Moreover, on the basis of Theorem 2 we have:

$$f(x) = P(x) + x^n\varepsilon_1(x),\ \ g(x) = Q(x) + x^n\varepsilon_2(x),$$
$$\text{where } \lim_{x\to 0}\varepsilon_1(x) = 0,\ \ \lim_{x\to 0}\varepsilon_2(x) = 0.$$

Hence,

$$f(x)\cdot g(x) = [P(x) + x^n\varepsilon_1(x)]\cdot[Q(x) + x^n\varepsilon_2(x)] =$$
$$= P(x)\cdot Q(x) + x^n[Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^m\varepsilon_1(x)\varepsilon_2(x)] =$$
$$= \varphi_n(P(x)Q(x)) + x^{n+1}S(x) + x^n[Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)].$$

Therefore, we get:

$$f(x)\cdot g(x) = \varphi_n(P(x)\cdot Q(x)) +$$
$$+ x^n[xS(x) + Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)],$$
$$\text{where:}\quad \lim_{x\to 0}[xS(x) + Q(x)\varepsilon_1(x) + P(x)\varepsilon_2(x) + x^n\varepsilon_1(x)\varepsilon_2(x)] = 0.$$

From Theorem 2 we assert that

$$(DL)_n^0(f\cdot g) = \varphi_n(P\cdot Q).$$

c) It follows from the assumption that $d^o(\chi) = n$. Moreover,

$$\lim_{x\to 0}\ \frac{1}{x^n}[\frac{f(x)}{g(x)} - \chi(x)] = \lim_{x\to 0}\ \frac{1}{x^n g(x)}[f(x) - \chi(x)g(x)] =$$

$$= \lim_{x\to 0}\ \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - \chi(x)(Q(x) + x^n\varepsilon_2(x))] =$$

$$= \lim_{x\to 0} \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - \chi(x)(Q(x)) - \chi(x)x^n\varepsilon_2(x)] =$$

$$= \lim_{x\to 0} \frac{1}{x^n g(x)}[P(x) + x^n\varepsilon_1(x) - (P(x) - x^{n+1}Q_n(x)) - \chi(x)x^n\varepsilon_2(x)] =$$

$$= \lim_{x\to 0} \frac{1}{g(x)}[\varepsilon_1(x) + xQ_n(x) - \chi(x)\varepsilon_2(x)] = 0.$$

Hence, $(DL)_n^0(\frac{f}{g}) = \chi.$

Theorem 5. (About finite expansion of composition of functions)

$$\left[(DL)_n^0(f) = P \wedge (DL)_m^0(g) = Q \wedge f(0) = 0 \wedge f \neq 0 \wedge m\cdot k \geqslant n\right] \Rightarrow$$
$$\Rightarrow (DL)_n^0(g\circ f) = \varphi_n(Q\circ P),$$

where k is the power of the lowest term of a polynomial P.

Proof.
It follows from the assumption and Theorem 2 that

$$f(x) = P(x) + x^n\varepsilon_1(x), \quad \text{where } \varepsilon_1(x) \underset{x\to 0}{\to} 0, \text{ hence } f(0) = P(0) = 0.$$

There exists

$$k > 0, \text{ such that } P(x) = x^k P_1(x).$$

Then we obtain:

$$f(x) = P(x) + x^n\varepsilon_1(x) = x^k P_1(x) + x^n\varepsilon_1(x) = x^k(P_1(x) + x^{n-k}\varepsilon_1(x)).$$

On the basis of Theorem 2, a function g has the following property:

$$g(y) = Q(y) + y^m\varepsilon_2(y), \quad \varepsilon_2(y) \underset{y\to 0}{\to} 0. \tag{7}$$

From (7) we get for the composition of functions $g \circ f$

$$(g \circ f)(x) = g(f(x)) = Q(f(x)) + [f(x)]^m \varepsilon_2(f(x)) =$$

$$= Q(f(x)) + [x^k(P_1(x) + x^{n-k}\varepsilon_1(x))]^m \cdot \varepsilon_2(f(x)).$$

In the interval $(f(x), P(x))$ we use Lagrange's mean value theorem for a polynomial Q

$$\frac{Q(f(x)) - Q(P(x))}{f(x) - P(x)} = Q'(C_x), \text{ where } C_x \in (f(x), P(x)),$$

therefore,

$$Q(f(x)) - Q(P(x)) = [f(x) - P(x)]Q'(C_x) = x^n \varepsilon_1(x) Q'(C_x).$$

A function $g \circ f$ gets the following value:

$$(g \circ f)(x) = Q(P(x)) + x^n \varepsilon_1(x) Q'(C_x) + x^{km}[P_1(x) + x^{n-k}\varepsilon_1(x)]^m \varepsilon_2(f(x)).$$

Transforming a polynomial $Q(P(x)$

$$Q(P(x)) = \varphi_n(Q(P(x)) + x^{n+1}S(x))$$

we obtain

$$(g \circ f)(x) = \varphi_n(Q(P(x)) + x^{n+1} \cdot S(x) + x^n \varepsilon_1(x) Q'(C_x) + x^{km}[P_1(x) + x^{n-k}\varepsilon_1]^m \varepsilon_2(f(x)).$$

Hence,

$$\lim_{x \to 0} \frac{1}{x^n}[(g \circ f)(x) - \varphi_n((Q \circ P)(x))] =$$

$$= \lim_{x \to 0} \{xS(x) + \varepsilon_1(x) Q'(x) + x^{km-n}[P_1(x) + x^{n-k}\varepsilon_1(x)]^m \varepsilon_2(f(x))\} = 0,$$

as $km - n \geqslant 0$.
On the basis of Definition 1a we assert that

$$(DL)^0_n(g \circ f) = \varphi_n(Q \circ P).$$

Corollary 1.

$$(DL)^0_n(f) = P \wedge (DL)^0_n(g) = Q \wedge f(0) = 0 \Rightarrow (DL)^0_n(g \circ f) = \varphi_n(Q \circ P).$$

This corollary follows immediately from Theorem 5 for $k = 1$ and $m = n$.

Theorem 6. (The Taylor–Maclaurin finite expansion.)
If a function f is differetiable n times on an open interval I containing zero, $f^{(n)}$ is continuous at zero and

$$P(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0), \quad \text{then} \quad (DL)_n^0(f) = P.$$

This Theorem follows from Theorem 2 and the Taylor–Maclaurin theorem. It should be noted that a finite expansion of a function f can exist, though the derivative $f^{(n)}(0)$ does not exist. For example, a function f defined as:

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x^3 \sin(\frac{1}{x}), & \text{if } x \neq 0, \end{cases}$$

does not have the derivative $f''(0)$, since

$$f'(x) = \begin{cases} 0, & \text{if } x = 0, \\ 3x^2 \sin(\frac{1}{x}) - x\cos(\frac{1}{x}), & \text{if } x \neq 0, \end{cases}$$

but the limit of

$$\frac{f'(x) - f'(0)}{x} = 3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

does not exist when $x \to 0$.

On the other hand, the zero polynomial O is a finite expansion of the second degree for the function f in the neighborhood of the point $x = 0$ as

$$d^o(0) = -\infty \leq 2 \quad \text{and} \quad \lim_{x\to 0} \frac{1}{x^2}[f(x) - O(x)] = \lim_{x\to 0}(x \sin(x)) = 0.$$

Theorem 7 (about integration and differentiation of finite expansions):

a) If a function f is continuous on an open interval I, $0 \in I$, $(DL)_n^0(f) = P$, f is the derivative of a function $g : I \to R$ *and* $Q(x) = g(0) + \int_0^x P(t)dt$, then $D_{n+1}^0(g) = Q$.

b) If $f^{(n)}$ is continuous on an open interval I, $0 \in I$ and $(DL)_n^0(f) = P$, then $D_{n-1}^0(f') = P'$.

Proof.
a) A continuous function f can be written as

$$f(x) = P(x) + x^n \varepsilon(x),$$

where a function

$$\varepsilon(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{x^n}(f(x) - P(x)), & \text{if } x \neq 0 \end{cases}$$

is continuous on the interval I. From the assumption we have $g'(x) = f(x)$ for $x \in I$, therefore

$$\int_0^x g'(t)\,dt = \int_0^x f(x)\,dt = \int_0^x P(t)\,dt + \int_0^x t^n \varepsilon(t)\,dt,$$

hence

$$g(x) - g(0) = \int_0^x P(t)\,dt + \int_0^x t^n \varepsilon(t)\,dt$$

or

$$g(x) - Q(x) = \int_0^x t^n \varepsilon(t)\,dt.$$

Let us show that

$$\lim_{x\to 0} \frac{1}{x^{n+1}}[g(x) - Q(x)] = \lim_{x\to 0} \frac{1}{x^{n+1}} \int_0^x t^n \varepsilon(t)\,dt = 0.$$

In fact, substituting $u = \frac{t}{x}$ in the last integral we obtain:

$$\frac{1}{x^{n+1}} \int_0^x t^n \varepsilon(t)\,dt = \frac{1}{x^{n+1}} \int_0^1 (ux)^n \varepsilon(ux) x\,du = \int_0^1 u^n \varepsilon(ux)\,du.$$

Let $\varepsilon_1(x) = \int_0^1 u^n \varepsilon(ux)\,du$. As a function $\varepsilon(x)$ is continuous at the point $x = 0$, i.e. the Cauchy condition is filfilled:

$$\underset{\omega>0}{\forall} \ \underset{\delta_\omega>0}{\exists} \ \underset{x}{\forall} (|x| < \delta_\omega \Rightarrow |\varepsilon(x)| < \omega),$$

hence, for $|x| < \delta_\omega$ we have $|\varepsilon(x)| < \omega$. Then for a function $\varepsilon_1(x)$ we obtain:

$$|\varepsilon_1(x)| = \left| \int_0^1 u^n \varepsilon(ux)\,du \right| \leqslant \int_0^1 u^n |\varepsilon(ux)|\,du \leqslant \int_0^1 \omega u^n\,du = \omega \left[\frac{u^{n+1}}{n+1} \right]_0^1 = \frac{\omega}{n+1}$$

or $\lim\limits_{x\to 0} \varepsilon_1(x) = 0$.

As

$$\lim_{x\to 0} \frac{1}{x^{n+1}}[g(x) - Q(x)] = \lim_{x\to 0} \int_0^1 u^n \varepsilon(ux)\,du = \lim_{x\to 0} \varepsilon_1(x) = 0$$

$$\text{and} \quad d^0(g) \leqslant n+1, \quad \text{then} \quad (DL)^0_{n+1}(g) = Q.$$

b) Under continuity assumption for a function $f^{(n)}$ we can use the Maclaurin formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + x^n\varepsilon(x)$$

On the other hand, on the basis of assumption $(DL)^0_n(f) = P$ we have $f(x) = P + x^n\varepsilon(x)$. Therefore,

$$P(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0).$$

Then $f'(x) = P'(x) + x^{n-1}\varepsilon_1(x)$, where $\varepsilon_1(x) \underset{x\to 0}{\to} 0$, and $d^o(P') \leqslant n-1$.

Hence, we get $(DL)^0_{n-1}(f') = P'$.

Example 2.

Evaluate the limit $\lim\limits_{x\to 0}\dfrac{e^x - \cos(x) - x}{x - \ln(1+x)}$ using the finite expansion.

Individual functions can be represented by the following finite expansions of the second degree (see Table 1):

$e^x = 1 + x + \frac{x^2}{2} + x^2\varepsilon_1(x)$,

$\cos(x) = 1 - \frac{x^2}{2} + x^2\varepsilon_2(x)$,

$\ln(1+x) = x - \frac{x^2}{2} + x^2\varepsilon_3(x)$,

$e^x - \cos(x) - x = x^2 + x^2\varepsilon_4(x)$,

$x - \ln(1+x) = \frac{x^2}{2} + x^2\varepsilon_5(x)$.

The functions $e^x - \cos(x) - x$, $x - \ln(1+x)$ have the finite expansions x^2 and $\frac{x^2}{2}$, respectively. Dividing the polynomial x^2 by the polynomial $\frac{x^2}{2}$ we obtain the polynomial 2. Therefore, it follows from Theorem 4c that

$$\frac{e^x - \cos(x) - x}{x - \ln(1+x)} = 2 + x^2\varepsilon(x), \text{ where } \varepsilon(x) \underset{x\to 0}{\to} 0,$$

$$\text{hence, } \lim_{x\to 0}\frac{e^x - \cos(x) - x}{x - \ln(1+x)} = 2.$$

Example 3.

Find the finite expansion of the 5th degree for the function

$$h(x) = \tan(x - \sin(x))$$

in the neighborhood of the point $x = 0$.

Let $f(x) = x - \sin(x)$, $g(y) = \tan(y)$.
As $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^6\varepsilon_1(x)$ (see Table 1),
then $f(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + x^6\varepsilon_2(x)$,
where $\varepsilon_2(x) \underset{x\to 0}{\longrightarrow} 0$. A function f has the following finite expansion of the 5th degree

$$P(x) = \frac{x^3}{3!} - \frac{x^5}{5!}.$$

The least power in the polynomial P is $k = 3$. Therefore, using the condition

$$m \cdot k \geqslant n = 5 \text{ (Theorem 5) we obtain } m \geqslant \frac{5}{3} \text{ or } m = 2.$$

For the function g we calculate the finite expansion of the second degree. As $\tan(y) = y + y^2\varepsilon(y)$ (Table 1), then this expansion is equal to $Q(y) = y$.

The composition of functions $h(x) = (g \circ f)(x) = g(f(x)) = g(x - \sin(x)) = \tan(x - \sin(x))$ has the following finite expansion of the 5th degree (Theorem 5):

$$\varphi_5((Q \circ P)(x)) = \varphi_5\left(Q(P(x))\right) = \varphi_5\left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) = \frac{x^3}{3!} - \frac{x^5}{5!}.$$

Example 4.
Find the finite expansion of the nth degree for the function $g(x) = \ln\frac{1}{1-x}$ using the finite expansion of the $(n-1)$th degree for the function $f(x) = \frac{1}{1-x}$. Using the finite expansion of the function g evaluate the limit

$$\lim_{x\to 0}(1-x)\ln\left(\frac{1}{1-x}\right).$$

Since

$$f(x) = g'(x) \quad \text{and} \quad f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots + x^{n-1} + x^{n-1}\varepsilon(x) \text{ (Table 1)},$$

we can use Theorem 7a.
The finite expansion of the nth degree for the function g is as follows:

$$Q(x) = g(0) + \int_0^x P(t)\,dt, \quad \text{where} \quad P(t) = 1 + t + t^2 + \ldots + t^{n-1}.$$

Then

$$Q(x) = \ln(1) + \int_0^x (1 + t + t^2 + \ldots + t^{n-1})\,dt =$$

$$= \left[t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots + \frac{t^n}{n}\right]_0^x = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n}.$$

The function g has the form:

$$g(x) = \ln(\frac{1}{1-x}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n} + x^n\varepsilon(x), \quad \text{where} \quad \lim_{x\to 0}\varepsilon_1(x) = 0.$$

The finite expansion of the second degree for the function $h(x) = (1-x)\ln\frac{1}{1-x}$ is:

$$\varphi_2\left[(1-x)\left(x + \frac{x^2}{2} + \ldots + \frac{x^n}{n}\right)\right] = x - \frac{1}{2}x^2.$$

Then

$$(1-x)\ln\frac{1}{1-x} = x + \frac{2}{3}x^2 + x^2\varepsilon_2(x), \quad \text{where} \quad \varepsilon_2(x) \underset{x\to 0}{\to} 0)$$

and $\lim_{x\to 0}(1-x)\ln\frac{1}{1-x} = 0.$

It is obvious that this limit can be easily evaluated, but we would like to show how to use Theorem 7a.

Table 1. Representation of several functions by finite expansions

No.	Function
1.	$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + x^n\varepsilon(x)$
2.	$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + x^n\varepsilon(x)$
3.	$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x)$
4.	$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x)$
5.	$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^6\varepsilon(x)$
6.	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x)$

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