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Formalization of the Sentential Logic Dual to Łukasiewicz's Three-valued Logic*

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Abstract

Dual logics with respect to Łukasiewicz's logics were investigated by G. Malinowski, M. Spasowski and R. Wójcicki in [4,5]. Our aim is to discuss the generalized method of natural deduction for the logic which is dual to Łukasiewicz's three-valued logic.

1. Introduction

The generalized method of natural deduction is based on a set of rules of decomposition, where, apart from rules of admission, we accept also some rules of rejection. We start with the empty set of axioms.
Proofs of tautologies are of the form of trees, whose branches are constructed by applying given rules of decomposition as follows.
If the rule is in the conjunctive form

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$$\begin{array}{c} s \\ \hline s_1 \\ s_2 \\ \vdots \\ s_n \end{array} \;,$$

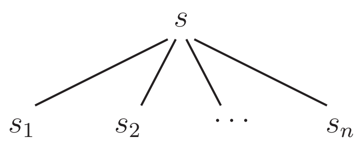
then we extend the branch containing the vertex s by adding n new vertices like this:

$$\begin{array}{c} s \\ | \\ s_1 \\ | \\ s_2 \\ | \\ \vdots \\ | \\ s_n \end{array}$$

If the rule is in the disjunctive form

$$\frac{s}{s_1 \quad s_2 \quad \cdots \quad s_n}\,,$$

then we put n new branches starting in the vertex s like this:



The method was described in details in [1].

Let $\mathbb{M} = (U, V, \mathbf{f})$ be a finite matrix of any propositional calculus based on a language $J = (S, \mathbf{F})$, where $\mathbf{F}$ is a finite set of functors and S is the set of all formulas of the language J.

Let $\alpha \in S$, $i \in V$, $j \in U \setminus V$. We introduce two kinds of signed expressions: $\vdash_i \alpha$ and $\dashv_j \alpha$, which are meant to denote the fact that α is admitted in the $i-\ th$ degree or α is rejected in the $j-th$ degree, respectively. If α is a propositional variable then expressions $\vdash_i \alpha$ and $\dashv_j \alpha$ are said to be *atomic.*

For every formula α and every $j \in U \setminus V$ of the language J we can construct the tree of decomposition for the signed expression $\dashv_j \alpha$.

[1] We start with the vertex signed by $\dashv_j \alpha$, where $j \in U \setminus V$ and apply the appropriate rule of decomposition adding one or more new branches as it was described above.

[2] In the $n-th$ step we apply the rules to all vertices (which are labeled by signed expressions) which were obtained in the $(n-1)-th$ step and which are not atomic.

[3] The construction is finished, where all leaves of the tree of decomposition contain either atomic signed expressions or expressions which cannot be decomposed by any rule.

- A branch of the tree of decomposition is closed if it contains two vertices labeled by atomic signed expressions of the form $*_i\alpha, *_j\alpha$ for some $i \neq j, i, j \in U$, where

  $$*_s\alpha = \begin{cases} \vdash_s \alpha & \text{for} \quad s \in V, \\ \dashv_s \alpha & \text{for} \quad s \in U \setminus V \end{cases}$$

  or it contains a vertex which is labeled by a non - atomic signed expression which cannot be decomposed by any rule.

- A tree of decomposition is closed iff every its branch is closed.

We will say that a formula α is admissible iff the tree of decomposition for the signed expression $\dashv_j \alpha$ is closed, for every $j \in U \setminus V$. Let us denote by T the set off all admissible formulas.

It was shown in [1] that the set tautologies of any finite matrix is identical to the set of all admissible formulas (in the sense of the generalized method of natural deduction, described above). It means that the method can be applied to examining if a given formula is a tautology of a sentential calculus with a finite matrix.

2. The generalized method of natural deduction for the logic $\mathcal{L}_3^d$.

Let $\mathcal{J} = (S, \{\Rightarrow, \neg, \wedge\})$ be the language of Łukasiewicz's sentential calculus, where connectives $\Rightarrow, \neg, \wedge$ stand for implication, negation and conjunction, respectively.

The Łukasiewicz's three - valued matrix has a form:

$$\mathbb{M}_3 = (\left\{0, \frac{1}{2}, 1\right\}, \{1\}, \{f_{\Rightarrow}, f_{\neg}, f_{\wedge}\}),$$

where functions $f_{\Rightarrow}, f_{\neg}, f_{\wedge}$ are defined as follows:

$$f_{\Rightarrow}(x, y) = min\{1, 1 - x + y\},$$

$$f_{\neg}(x) = 1 - x,$$

$$f_{\wedge}(x, y) = min\{x, y\}.$$

The disjunction functor $\vee$ can be defined by:

$$\alpha \vee \beta \overset{df}{=} \neg(\neg\alpha \wedge \neg\beta).$$

Additionally, let us define the functor $\overset{d}{\longrightarrow}$ as follows:

$$\alpha \overset{d}{\longrightarrow} \beta \overset{df}{=} \neg_d(\alpha \rightarrow \beta), \text{ where}$$

$$\neg_d\alpha \overset{df}{=} \alpha \Rightarrow (\alpha \wedge \neg\alpha),$$

$$\alpha \rightarrow \beta \overset{df}{=} \alpha \Rightarrow (\alpha \Rightarrow \beta).$$

The logic $\mathcal{L}_3^d$ dual to Łukasiewicz's three - valued logic can be characterized by the following matrix dual to the matrix $\mathbb{M}_3$:

$$\mathbb{M}_3^d = \left(\left\{ 0, \frac{1}{2}, 1 \right\}, \left\{ 0, \frac{1}{2} \right\}, \{ f_{\Rightarrow}, f_{\neg}, f_{\wedge} \} \right),$$

where functions $f_{\Rightarrow}, f_{\neg}, f_{\wedge}$ are defined in the same way as in the matrix $\mathbb{M}_3$.

We showed in [2] some axiomatization of the sentential logic $\mathcal{L}_3^d$.

We can take for the logic $\mathcal{L}_3^d$ the following rules of decomposition:

Negation $\neg$

$$\frac{\vdash_0 \neg\alpha}{\dashv \alpha}, \qquad \frac{\vdash_{\frac{1}{2}} \neg\alpha}{\vdash_{\frac{1}{2}} \alpha}, \qquad \frac{\dashv \neg\alpha}{\vdash_0 \alpha}.$$

Implication $\Rightarrow$

$$\frac{\vdash_0 \alpha \Rightarrow \beta}{\begin{array}{c}\dashv \alpha \\ \vdash_0 \beta\end{array}}, \qquad \frac{\vdash_{\frac{1}{2}} \alpha \Rightarrow \beta}{\begin{array}{cc}\vdash_{\frac{1}{2}} \alpha & \dashv \alpha \\ \vdash_0 \beta & \vdash_{\frac{1}{2}} \beta\end{array}}, \qquad \frac{\dashv \alpha \Rightarrow \beta}{\begin{array}{ccc}\vdash_0 \alpha & \vdash_{\frac{1}{2}} \alpha & \dashv \beta \\ & \vdash_{\frac{1}{2}} \beta & \end{array}}.$$

Conjunction $\wedge$

$$\frac{\vdash_0 \alpha \wedge \beta}{\vdash_0 \alpha \qquad \vdash_0 \beta}, \qquad \frac{\vdash_{\frac{1}{2}} \alpha \wedge \beta}{\begin{array}{ccc}\vdash_{\frac{1}{2}} \alpha & \vdash_{\frac{1}{2}} \alpha & \dashv \alpha \\ \vdash_{\frac{1}{2}} \beta & \dashv \beta & \vdash_{\frac{1}{2}} \beta\end{array}}, \qquad \frac{\dashv \alpha \wedge \beta}{\begin{array}{c}\dashv \alpha \\ \dashv \beta\end{array}}.$$

Disjunction $\vee$

$$\frac{\vdash_0 \alpha \vee \beta}{\begin{array}{c}\vdash_0 \alpha \\ \vdash_0 \beta\end{array}}, \qquad \frac{\vdash_{\frac{1}{2}} \alpha \vee \beta}{\begin{array}{ccc}\vdash_{\frac{1}{2}} \alpha & \vdash_{\frac{1}{2}} \alpha & \vdash_0 \alpha \\ \vdash_0 \beta & \vdash_{\frac{1}{2}} \beta & \vdash_{\frac{1}{2}} \beta\end{array}}, \qquad \frac{\dashv \alpha \vee \beta}{\dashv \alpha \qquad \dashv \beta}.$$

Dual implication $\xrightarrow{d}$

$$\frac{\vdash_0 \alpha \xrightarrow{d} \beta}{\vdash_0 \alpha \qquad \vdash_{\frac{1}{2}} \alpha \qquad \dashv \beta}\ , \qquad \frac{\dashv \alpha \xrightarrow{d} \beta}{\begin{array}{cc} \dashv \alpha & \dashv \alpha \\ \vdash_0 \beta & \vdash_{\frac{1}{2}} \beta \end{array}}\ ,$$

where symbols $\vdash_0$, $\vdash_{\frac{1}{2}}$, $\dashv$ denote:
$\vdash_0 \alpha$ - formula α is admitted in the 0 degree,
$\vdash_{\frac{1}{2}} \alpha$ - formula α is admitted in the $\frac{1}{2}$ degree,
$\dashv \alpha$ - formula α is rejected, in other words $\dashv \alpha \Leftrightarrow \dashv_1 \alpha$.

Formulas of the logic $\mathcal{L}_3^d$ preceded by one of the following symbols $\vdash_0$, $\vdash_{\frac{1}{2}}$, $\dashv$ will be called signed expressions. We can build the tree of decomposition for any signed expression $\dashv \alpha$, where α is a formula of $\mathcal{L}_3^d$.

It can be observed that a branch of the decomposition tree is closed iff either a signed expression

$$\vdash_{\frac{1}{2}} \gamma \xrightarrow{d} \delta$$

or one of the pairs of signed expressions :

$$\left\{ \begin{array}{l} \vdash_0 p \\ \vdash_{\frac{1}{2}} p \end{array} \right. , \qquad \left\{ \begin{array}{l} \vdash_0 p \\ \dashv p \end{array} \right. , \qquad \left\{ \begin{array}{l} \vdash_{\frac{1}{2}} p \\ \dashv p \end{array} \right. ,$$

where p is a propositional variable, appears in it. A closed branch will be marked by (c).
A tree of decomposition is closed if all of its branches are closed.

Let us denote by T^* the set of all admissible formulas of $\mathcal{L}_3^d$. Thus T^* is the set of all $\alpha \in S$ for which the tree of decomposition for a signed expression $\dashv \alpha$ is closed.

Since, the matrix of the logic $\mathcal{L}_3^d$ is finite, therefore, by [1], the following theorem holds

Theorem 1.

$$\mathrm{T}^* = E(\mathbb{M}_3^d),$$

where $E(\mathbb{M}_3^d)$ denotes the content of the matrix $\mathbb{M}_3^d$.

Here are some examples showing how to apply the method.

Example 1.

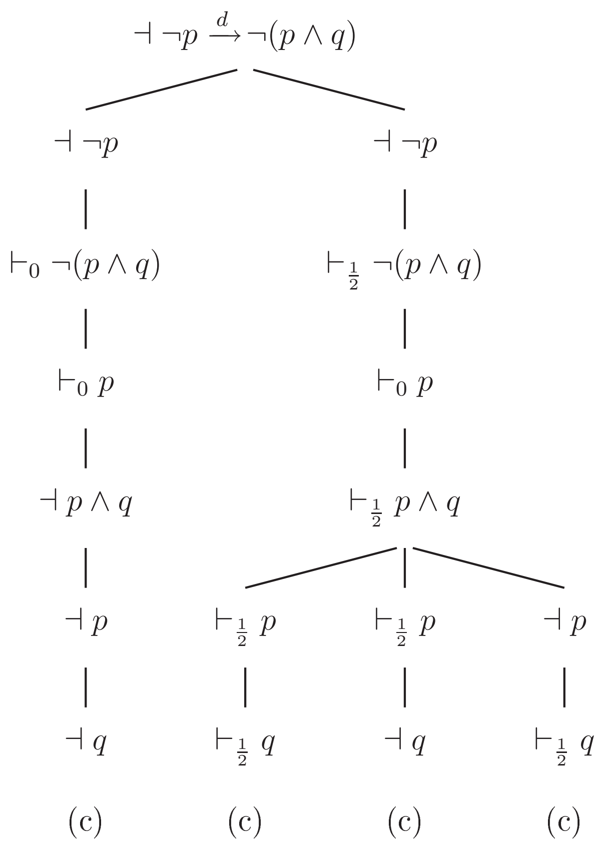
We will use the generalized method of natural deduction to prove that

[1] $\neg p \xrightarrow{d} \neg(p \wedge q)$,

[2] $p \xrightarrow{d} (p \xrightarrow{d} (q \xrightarrow{d} p))$

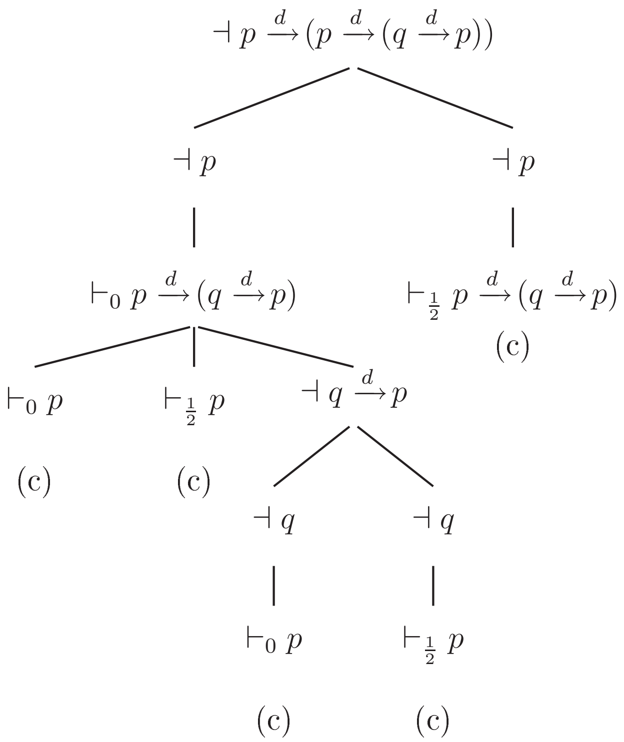
are tautologies of $\mathcal{L}_3^d$.

1. We construct the tree of decomposition for the signed expression $\dashv \neg p \xrightarrow{d} \neg(p \wedge q)$.



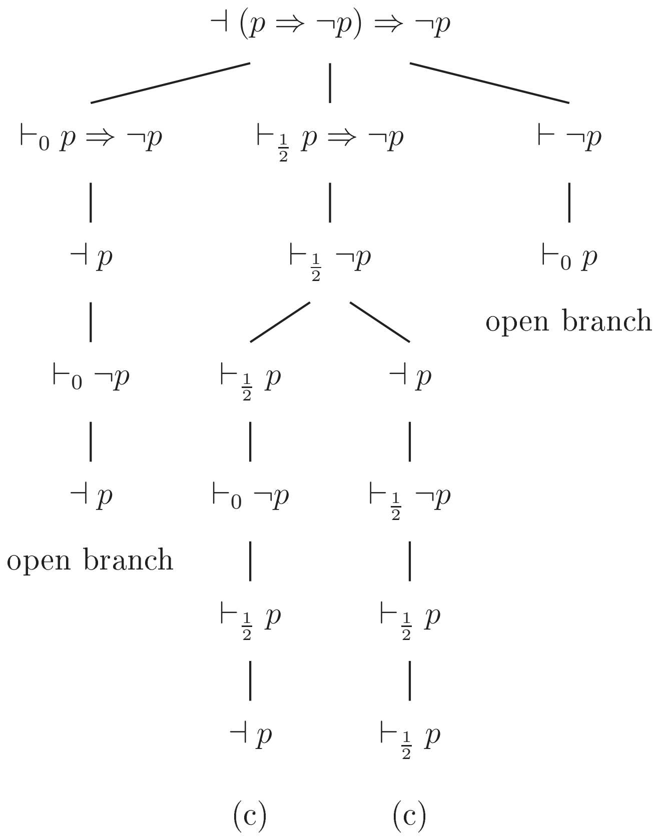
2. We construct the tree of decomposition for the signed expression

$$\dashv p \xrightarrow{d} (p \xrightarrow{d} (q \xrightarrow{d} p)).$$



Example 2.

Now, we are going to build the tree of decomposition for the signed expression $\dashv (p \Rightarrow \neg p) \Rightarrow \neg p$ to show that the formula $(p \Rightarrow \neg p) \Rightarrow \neg p$ isn't a tautology in $\mathcal{L}_3^d$.



It is possible to formulate a formal algorithm for constructing the tree of decomposition and checking if it is closed.

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