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Queueing Systems with Common Memory Space*

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Abstract

In the present paper we investigate queueing systems of different types with customers having some random space requirements, connected via common memory space. For such systems combinations we determine stationary loss probability and the distribution of customers present in each system.

1 Introduction

Queueing systems with non-homogeneous customers have been used to model and solve the various practical problems occurring in the design of computer or communicating systems [1–3]. The above non-homogeneity means that each customer (independently of others) has some space requirement ζ, and the service time ξ of the customer generally depends on his space requirement. So, the non-negative random variables ζ and ξ are generally dependent.

Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the distribution function of the random vector (ζ, ξ). The space is occupied by the customer at the

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epoch he arrives and is released entirely at the epoch he completes service.

Denote as $\sigma(t)$ the sum of space requirements of all customers present in the system at time moment t. The process $\sigma(t)$ is called the total (customers) volume. In real systems the values, that the random process $\sigma(t)$ can take, are limited by some constant value $V > 0$, which is called the memory volume of the system. The limitation of the total volume leads to additional losses of customers. In fact, a customer having the space requirement x, who arrives at the epoch τ, when there are idle servers or waiting positions, will be admitted to the system, if $\sigma(\tau^-) + x \le V$. Otherwise (if $\sigma(\tau^-) + x > V$) the customer will be lost.

Different queueing systems with limited memory volume were analyzed in the papers [4–10]. In the present paper we investigate combinations of (may be different) queueing systems with non-homogeneous customers and common limited memory space. It iss clear, that such models can be used in computer and communicating networks designing.

Consider, for example, two independent classical $M/M/n/m$ queues, denoting as $M/M/n_i/m_i$, where $i = 1, 2$, $1 \le n_i \le \infty$, $0 \le m_i \le \infty$. Let a_i, μ_i be the rate of customer arrival process and service time parameter of the ith queue respectively. Now we additionally suppose, that each customer of ith queue has some space requirement ζ_i. Denote as $L_i(x)$ the distribution function of ζ_i random variable and as $\sigma(t)$ the sum of space requirements of all customers present in the both systems at time moment t. Suppose, that service time of the customer not depends on his space requirement and the both queues have the common memory space, which is limited by the memory volume V. Due to the last supposition the queues under consideration become obviously dependent.

We first determine the stationary customers number distribution and loss probability for each queue from the considered combination. Next we shall analyze a combination of $M/M/n/m$ queue and processor-sharing system with service time dependent on customer space requirement.

2 Process and characteristics

Let $\eta_i(t)$ be the number of customers present in ith queue at time moment t, $i = 1, 2$. Denote as $\sigma_j^i(t)$ the space occupied by jth customer of ith queue at the moment t, $j = \overline{1, \eta_i(t)}$. Then the combination of queues under consideration may be described by the next markovian random process

$$\left(\eta_1(t); \eta_2(t); \sigma_j^1(t), j = \overline{1, \eta_1(t)}; \sigma_j^2(t), j = \overline{1, \eta_2(t)}\right), \tag{1}$$

where $\sum\limits_{i=1}^{2} \sum\limits_{j=1}^{\eta_i(t)} \sigma_j^i(t) = \sigma(t)$.

The process (1) we shall characterize by the following functions.

$$G(k_1, k_2, x, t) = \mathbf{P}\{\eta_1(t) = k_1, \eta_2(t) = k_2, \sigma(t) < x\}; \tag{2}$$

$$P(k_1, k_2, t) = \mathbf{P}\{\eta_1(t) = k_1, \eta_2(t) = k_2\} = G(k_1, k_2, V, t), \tag{3}$$

$$k_i = \overline{0, n_i + m_i},\ k_1 + k_2 \geq 1,\ i = 1, 2;$$

$$P_0(t) = P(0, 0, t) = \mathbf{P}\{\eta_1(t) = 0, \eta_2(t) = 0\}. \tag{4}$$

It's clear, that steady state conditions for the model under consideration are always satisfied, if V is finite. Then $\eta_i(t) \Rightarrow \eta_i$, $\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergence, where η_i, σ are the stationary number of customers present in ith queue and stationary customers total volume respectively. So, the following limits exist

$$g(k_1, k_2, x) = \lim_{t\to\infty} G(k_1, k_2, x, t) = \mathbf{P}\{\eta_1 = k_1, \eta_2 = k_2, \sigma < x\}; \tag{5}$$

$$p(k_1, k_2) = \lim_{t\to\infty} P(k_1, k_2, t) = \mathbf{P}\{\eta_1 = k_1, \eta_2 = k_2\} = g(k_1, k_2, V), \tag{6}$$

$$k_i = \overline{0, n_i + m_i},\ k_1 + k_2 \geq 1,\ i = 1, 2;$$

$$p_0 = p(0, 0) = \lim_{t\to\infty} P_0(t) = \mathbf{P}\{\eta_1 = 0, \eta_2 = 0\}. \tag{7}$$

3 Stationary customers number distribution

Let $\delta_{i,k}$ be Kronecker's symbol: $\delta_{i,k} = \begin{cases} 0, & i \neq k, \\ 1, & i = k. \end{cases}$ Further we shall use the next notations: $\kappa_i = \min(k_i, n_i)$, $\gamma_i = \min(k_i + 1, n_i)$. It can be easily shown by means of process (1) analysis, that the functions (2)–(4) satisfy the following differential equations:

$$P_0'(t) = -\left(a_1 L_1(V) + a_2 L_2(V)\right) P_0(t) + \mu_1 P(1,0,t) + \mu_2 P(0,1,t); \quad (8)$$

$$P'(0,1,t) = a_2 P(0,0,t) L_2(V) - a_1 \int_0^V G(0,1,V-x,t)\, dL_1(x) -$$
$$-a_2 \int_0^V G(0,1,V-x,t)\, dL_2(x) - \mu_2 P(0,1,t) +$$
$$+\mu_1 P(1,1,t) + 2\mu_2 P(0,2,t); \quad (9)$$

$$P'(1,0,t) = a_1 P(0,0,t) L_1(V) - a_1 \int_0^V G(1,0,V-x,t)\, dL_1(x) -$$
$$-a_2 \int_0^V G(1,0,V-x,t)\, dL_2(x) - \mu_1 P(1,0,t) +$$
$$+2\mu_1 P(2,0,t) + \mu_2 P(1,1,t); \quad (10)$$

$$P'(k_1,k_2,t) = \left(1 - \delta_{0,k_1}\right) a_1 \int_0^V G(k_1 - 1, k_2, V-x,t)\, dL_1(x) +$$
$$+ \left(1 - \delta_{0,k_2}\right) a_2 \int_0^V G(k_1, k_2 - 1, V-x,t)\, dL_2(x) -$$
$$- \left(1 - \delta_{m_1+n_1,k_1}\right) a_1 \int_0^V G(k_1, k_2, V-x,t)\, dL_1(x) -$$
$$- \left(1 - \delta_{m_2+n_2,k_2}\right) a_2 \int_0^V G(k_1, k_2, V-x,t)\, dL_2(x) -$$
$$-(\kappa_1\mu_1 + \kappa_2\mu_2) P(k_1,k_2,t) + \left(1 - \delta_{m_1+n_1,k_1}\right) \gamma_1 \mu_1 P(k_1+1,k_2,t) +$$
$$+ \left(1 - \delta_{m_2+n_2,k_2}\right) \gamma_2 \mu_2 P(k_1, k_2+1, t),$$
$$k_i = \overline{0, m_i + n_i},\ k_1 + k_2 \geq 2,\ i = 1,2. \quad (11)$$

If steady state conditions take place, from equations (8)–(11) when $t \to \infty$ we obtain the following equations for the functions (5)–(7):

$$0 = -\left(a_1 L_1(V) + a_2 L_2(V)\right) p_0 + \mu_1 p(1,0) + \mu_2 p(0,1); \quad (12)$$

$$0 = a_2 p(0,0) L_2(V) - a_1 \int_0^V g(0,1,V-x)\, dL_1(x) -$$
$$-a_2 \int_0^V g(0,1,V-x)\, dL_2(x) - \mu_2 p(0,1) + \mu_1 p(1,1) + 2\mu_2 p(0,2); \quad (13)$$

$$0=a_1p(0,0)L_1(V)-a_1\int_0^V g(1,0,V-x)dL_1(x)-$$

$$-a_2\int_0^V g(1,0,V-x)dL_2(x)-\mu_1p(1,0)+2\mu_1p(2,0)+\mu_2p(1,1); \quad (14)$$

$$0=\left(1-\delta_{0,k_1}\right)a_1\int_0^V g(k_1-1,k_2,V-x)dL_1(x)+$$

$$+\left(1-\delta_{0,k_2}\right)a_2\int_0^V g(k_1,k_2-1,V-x)dL_2(x)-$$

$$-\left(1-\delta_{m_1+n_1,k_1}\right)a_1\int_0^V g(k_1,k_2,V-x)dL_1(x)-$$

$$-\left(1-\delta_{m_2+n_2,k_2}\right)a_2\int_0^V g(k_1,k_2,V-x)dL_2(x)-$$

$$-(\kappa_1\mu_1+\kappa_2\mu_2)p(k_1,k_2)+\left(1-\delta_{m_1+n_1,k_1}\right)\gamma_1\mu_1p(k_1+1,k_2)+$$

$$+\left(1-\delta_{m_2+n_2,k_2}\right)\gamma_2\mu_2p(k_1,k_2+1),$$

$$k_i=\overline{0,m_i+n_i},\ k_1+k_2\geq 2,\ i=1,2. \quad (15)$$

Denote the Stieltjes convolution of the ith order of the function $L_i(x)$ as $L_i^{(k)}(x)$, i.e.

$$L_i^{(0)}(x)\equiv 1,\quad L_i^{(k)}(x)=L_i^{(k-1)}*L_i(x)=$$

$$=\int_0^x L_i^{(k-1)}(x-u)dL_i(u),\ k=1,2,\ldots.$$

We shall use the next notation for the Stieltjes convolution of functions $F_i(x),\ i=\overline{1,r}$:

$$F_1*\cdots*F_r(x)=\mathop{*}_{i=1}^{r}F_i(x)$$

We also introduce the following notation:

$$N_i(k)=\begin{cases}\dfrac{(n_i\rho_i)^k}{k!}, & k=\overline{0,n_i},\\ \dfrac{n_i^{n_i}\rho_i^k}{n_i!}, & k=\overline{n_i+1,n_i+m_i},\end{cases} \quad (16)$$

where $\rho_i=a_i/(n_i\mu_i)$.

We can easy verify by direct substitution, that the solution of the equations (12)–(15) can be represented as

$$g(k_1,k_2,x)=p_0N_1(k_1)N_2(k_2)L_1^{(k_1)}*L_2^{(k_2)}(x),\ k_1+k_2\geq 1, \quad (17)$$

whence we obtain

$$p(k_1,k_2) = g(k_1,k_2,V) =$$
$$= p_0 N_1(k_1) N_2(k_2) L_1^{(k_1)} * L_2^{(k_2)}(V),\ k_i = \overline{0, m_i + n_i}. \tag{18}$$

From the normalization condition $\sum\limits_{k_1=0}^{n_1+m_1} \sum\limits_{k_2=0}^{n_2+m_2} p(k_1,k_2) = 1$ we also have that

$$p_0 = \left[\sum_{k_1=0}^{n_1+m_1} \sum_{k_2=0}^{n_2+m_2} N_1(k_1) N_2(k_2) L_1^{(k_1)} * L_2^{(k_2)}(V) \right]^{-1}. \tag{19}$$

4 The loss probability

The stationary loss probability p_L^i for customers of ith queue ($i = 1,2$) can be obtained from the following equilibrium equation [2], which will be written out for the case of $i = 1$:

$$a_1 \left(1 - p_L^1\right) = \mu_1 \sum_{k_1=1}^{n_1-1} k_1 p_{k_1}^1 + n_1 \mu_1 \left(1 - \sum_{k_1=1}^{n_1-1} p_{k_1}^1\right), \tag{20}$$

where $p_{k_1}^1 = \sum\limits_{k_2=0}^{n_2+m_2} p(k_1,k_2)$, $p_{k_2}^2 = \sum\limits_{k_1=0}^{n_1+m_1} p(k_1,k_2)$

It follows from (20) and analogous equation for $i = 2$ that

$$p_L^i = 1 - (n_i \rho_i)^{-1} \sum_{k_i=1}^{n_i-1} k_i p_{k_i}^i - {\rho_i}^{-1} \left(1 - \sum_{k_i=0}^{n_i-1} p_{k_i}^i\right). \tag{21}$$

5 Arbitrary number of queues with common memory space

In the present section we consider an evident generalization of analyzed combination of queues. Suppose, that we have a combination consisting of r, $r = 1,2,\ldots$, $M/M/n/m$ queues with non-homogeneous customers connected via common memory space. Let a_i, μ_i be the rate of customer arrival process and service time parameter of the ith queue respectively, $i = \overline{1,r}$, $L_i(x)$ be the distribution function of ith queue customer space requirement, V be the memory volume. Then for stationary probabilities $p(k_1,\dots,k_r) = \mathbf{P}\{\eta_1 = k_1,\dots,\eta_r = k_r\}$, that there are $k_1,\dots,k_r$ customers in corresponding queue, we obtain the following relation:

$$p(k_1,\dots,k_r) = p_0 \prod_{i=1}^{r} N_i(k_i) \mathop{*}_{i=1}^{r} L_i^{(k_i)}(V),$$

where the function $N_i(k)$ is determined by (16) relation, in which $i = \overline{1,r}$. The relation (19) for p_0 in this case has the next form:

$$p_0 = \left[\sum_{k_1,\dots,k_r} \prod_{i=1}^{r} N_i(k_i) \mathop{*}_{i=1}^{r} L_i^{(k_i)}(V) \right]^{-1},$$

where the summation for each subscript is carried out from 0 to n_i+m_i.

The loss probability of ith queue customer is determined by relation (21), where

$$p_{k_i}^i = \sum_{k_1,\dots,k_{i-1},k_{i+1},\dots,k_r} p(k_1,\dots,k_{i-1},k_i,k_{i+1},\dots,k_r),\ k = \overline{0, n_i+m_i}.$$

6 The case of single-server queues with common memory space

In the present section we shall analyze the case of two queues connected via common memory space ($r = 2$). The generalization for arbitrary r is obvious. If $n_i = 1$, we obtain from (16), that $N_i(k) = {\rho_i}^k$, $i = 1, 2$. Then, as it follows from (18)–(21),

$$p_0 = \left[\sum_{k_1=0}^{m_1+1} \sum_{k_2=0}^{m_2+1} {\rho_1}^{k_1} {\rho_2}^{k_2} L_1^{(k_1)} * L_2^{(k_2)}(V) \right]^{-1}, \tag{22}$$

$$p(k_1,k_2) = p_0 {\rho_1}^{k_1} {\rho_2}^{k_2} L_1^{(k_1)} * L_2^{(k_2)}(V),\ k_i = \overline{0, m_i+1}, \tag{23}$$

$$p_L^i = 1 - \frac{1}{\rho_i}\left(1 - p_0^i\right),\ i = 1, 2. \tag{24}$$

The obtained relations are generally not convenient for calculations because of Stieltjece convolutions presence. These convolutions can be generally estimated by Monte Carlo technique. They can precisely calculated in some special cases. For example, let ith queue customer space requirement has gamma distribution with the following density

$$l_i(x) = \gamma_f(\alpha_i, x) = \frac{f}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-fx},\ \alpha_i > 0,\ f > 0,\ i = 1, 2$$

(the parameter f is supposed the same for both queues). It's known, that in this case the convolution $L_1^{(k_1)} * L_2^{(k_2)}(x)$ is the gamma distribution function $Y_f(\alpha, x)$ with parameters f and $\alpha = k_1\alpha_1 + k_2\alpha_2$, i.e. we have

$$p(k_1, k_2) = p_0 {\rho_1}^{k_1} {\rho_2}^{k_2} Y_f(k_1\alpha_1 + k_2\alpha_2, V),$$

$$p_0 = \left[\sum_{k_1=0}^{m_1+1} \sum_{k_2=0}^{m_2+1} {\rho_1}^{k_1} {\rho_2}^{k_2} Y_f(k_1\alpha_1 + k_2\alpha_2, V) \right]^{-1}.$$

In this case the relation for the first queue customer loss probability has the following form:

$$p_L^1 = 1 - \frac{1}{\rho_1} \left[1 - \sum_{k_2=0}^{m_2+1} {\rho_2}^{k_2} Y_f(k_2\alpha_2, V) \right].$$

7 Some special cases

In this section we consider the case of $L_1(x) = L_2(x) = L(x)$ (the customers space requirements distributions are the same for both queues). Then from (18), (19) relations we obtain

$$p(k_1, k_2) = p_0 N_1(k_1) N_2(k_2) L^{(k_1+k_2)}(V),\ k_i = \overline{0, m_i + n_i},\ i = 1, 2;$$

$$p_0 = \left[\sum_{k_1=0}^{n_1+m_1} \sum_{k_2=0}^{n_2+m_2} N_1(k_1) N_2(k_2) L^{(k_1+k_2)}(V) \right]^{-1}.$$

Now suppose additionally that $n_1 = n_2 = 1$, $m_1 = m_2 = \infty$ and customer space requirement has an exponential distribution with the parameter f: $L(x) = 1 - e^{-fx}$. Suppose also, that $\rho_1 \neq 1$, $\rho_2 \neq 1$ and $\rho_1 \neq \rho_2$. Then from (22), (23) relations we obtain

$$p(k_1, k_2) = p_0 {\rho_1}^{k_1} {\rho_2}^{k_2} \left[1 - e^{-fV} \sum_{j=0}^{k_1+k_2-1} \frac{(fV)^j}{j!} \right],$$

$$k_i = 0, 1, \ldots,\ i = 1, 2;$$

$$p_0 = \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} {\rho_1}^{k_1} {\rho_2}^{k_2} \left[1 - e^{-fV} \sum_{j=0}^{k_1+k_2-1} \frac{(fV)^j}{j!} \right] \right\}^{-1}. \qquad (25)$$

Let us calculate the sum

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} {\rho_1}^{k_1} {\rho_2}^{k_2} \left[1 - e^{-fV} \sum_{j=0}^{k_1+k_2-1} \frac{(fV)^j}{j!} \right] =$$

$$= e^{-fV} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} {\rho_1}^{k_1} {\rho_2}^{k_2} \left[e^{fV} - \sum_{j=0}^{k_1+k_2-1} \frac{(fV)^j}{j!} \right] =$$

$$= e^{-fV} \sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \sum_{k_1=0}^{j} {\rho_1}^{k_1} \sum_{k_2=0}^{j-k_1} {\rho_2}^{k_2} =$$

$$= \frac{e^{-fV}}{1-\rho_2} \sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \sum_{k_1=0}^{j} {\rho_1}^{k_1} \left(1 - {\rho_2}^{j-k_1+1} \right), \qquad (26)$$

where

$$\sum_{k_1=0}^{j} {\rho_1}^{k_1} {\rho_2}^{j-k_1+1} = {\rho_2}^{j+1} \sum_{k_1=0}^{j} \left(\frac{\rho_1}{\rho_2} \right)^{k_1} = \frac{\rho_2}{\rho_2 - \rho_1} \left({\rho_2}^{j+1} - {\rho_1}^{j+1} \right),$$

$$\sum_{k_1=0}^{j} {\rho_1}^{k_1} = \frac{1 - {\rho_1}^{j+1}}{1-\rho_1},$$

whence

$$\sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \sum_{k_1=0}^{j} {\rho_1}^{k_1} = \sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \cdot \frac{1 - {\rho_1}^{j+1}}{1-\rho_1} = \frac{e^{fV} - \rho_1 e^{\rho_1 fV}}{1-\rho_1},$$

$$\sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \sum_{k_1=0}^{j} {\rho_1}^{k_1} {\rho_2}^{j-k_1+1} = \frac{\rho_2}{\rho_2 - \rho_1} \sum_{j=0}^{\infty} \frac{(fV)^j}{j!} \left({\rho_2}^{j+1} - {\rho_1}^{j+1} \right) =$$

$$= \frac{\rho_2 \left(\rho_2 e^{\rho_2 fV} - \rho_1 e^{\rho_1 fV} \right)}{\rho_2 - \rho_1}.$$

By substitution of calculated sums to relation (26) we obtain after simple transformations

$$p_0 = \left[\frac{1}{(1-\rho_1)(1-\rho_2)} - \frac{{\rho_1}^2 e^{-(1-\rho_1)fV}}{(1-\rho_1)(\rho_1-\rho_2)} - \frac{{\rho_2}^2 e^{-(1-\rho_2)fV}}{(1-\rho_2)(\rho_2-\rho_1)} \right]^{-1}. \qquad (27)$$

The loss probability can be determined from the relation (24), where for $i = 1$ we have

$$p_0^1 = \sum_{j=0}^{\infty} p(0,j) = \sum_{j=0}^{\infty} p_0 {\rho_2}^j \left[1 - e^{-fV} \sum_{i=0}^{j-1} \frac{(fV)^i}{i!}\right] =$$

$$= p_0 e^{-fV} \sum_{j=0}^{\infty} {\rho_2}^j \left[e^{fV} - \sum_{i=0}^{j-1} \frac{(fV)^i}{i!}\right] = p_0 e^{-fV} \sum_{j=0}^{\infty} {\rho_2}^j \sum_{i=j}^{\infty} \frac{(fV)^i}{i!} =$$

$$= p_0 e^{-fV} \sum_{i=0}^{\infty} \frac{(fV)^i}{i!} \sum_{j=0}^{i} {\rho_2}^j = p_0 e^{-fV} \sum_{i=0}^{\infty} \frac{(fV)^i}{i!} \cdot \frac{1 - {\rho_2}^{j+1}}{1 - \rho_2} =$$

$$= \frac{p_0 e^{-fV}}{1 - \rho_2} \left(e^{fV} - \rho_2 e^{\rho_2 fV}\right),$$

whence

$$p_L^1 = 1 - \frac{1}{\rho_1} \left[1 - \frac{p_0 \left(1 - \rho_2 e^{-(1-\rho_2)fV}\right)}{1 - \rho_2}\right] \tag{28}$$

and analogously

$$p_L^2 = 1 - \frac{1}{\rho_2} \left[1 - \frac{p_0 \left(1 - \rho_1 e^{-(1-\rho_1)fV}\right)}{1 - \rho_1}\right].$$

The relations (27), (28) can be generalized for the case of arbitrary r (where $\rho_i \neq \rho_j$ for $i \neq j$ and $\rho_i \neq 1$ for $i = \overline{1,r}$). The relation (27) in this case have the form

$$p_0 = \left[\prod_{i=1}^{r} \frac{1}{1 - \rho_i} - \sum_{i=1}^{r} \frac{{\rho_i}^r e^{-(1-\rho_i)fV}}{(1 - \rho_i) \prod_{j \neq i} (\rho_i - \rho_j)}\right]^{-1}.$$

The loss probability p_L^i is determined by the relation (24), where $i = \overline{1,r}$ and

$$p_0^i = p_0 \left[\prod_{j \neq i} \frac{1}{1 - \rho_j} - \sum_{j \neq i} \frac{{\rho_j}^{r-1} e^{-(1-\rho_j)fV}}{(1 - \rho_j) \prod_{k \neq j, k \neq i} (\rho_j - \rho_k)}\right].$$

8 M/M/n/m and processor sharing systems connected via common memory space

In present section we consider more complicated combination of two queueing systems connected via common memory space. The first of them is $M/M/n/m$ queue with non-homogeneous customers, which was represented in section 1. Let a_1, μ be the rate of customer arrival process and service time parameter of this queue respectively and $L_1(x)$ be the distribution function of customer space requirement ζ_1 of the queue. The second one is processor-sharing system [12]. Denote as ξ_1 and ξ_2 the service time of the first and second system respectively. Let ζ_2 be the customer space requirement of the second system and $F(x,t) = \mathbf{P}\{\zeta_2 < x, \xi_2 < t\}$ be the distribution function of the random vector (ζ_2, ξ_2). So, in second system the space requirement and service time are generally dependent and $L_2(x) = F(x,\infty)$, $B_2(t) = F(\infty,t)$ are the distribution functions of random variables ζ_2 and ξ_2 respectively. Let r be a maximum number of customers present in the second system (it is possible, that $r=\infty$). Suppose, that all customers present in the second system are numbered in random order [1]. Let $\eta_1(t)$, $\eta_2(t)$ be the number of customers present in the first and second system at time moment t.

Denote as $\xi_*^j(t)$ the length of time interval from the moment t to the epoch of jth customer service completion (in the second system). Let $\sigma_j^{(i)}(t)$ be the memory space occupied by jth customer of ith system, $i = 1, 2$. Then the combination of queues under consideration may be described by the next markovian random process

$$\left(\eta_1(t),\, \eta_2(t);\, \sigma_j^{(i)}(t),\, j = \overline{1, \eta_i(t)},\, i = 1,2;\, \xi_*^j(t),\, j = \overline{1, \eta_2(t)}\right). \quad (29)$$

Denote the total customers volume in the combination of the systems at the moment t as $\sigma(t) = \sum\limits_{i=1}^{2} \sum\limits_{j=1}^{\eta_i(t)} \sigma_j^{(i)}(t)$. Let V be the memory volume of the combination.

Below we shall use the following notation for vectors:

$$Y_k = (y_1, \ldots, y_k) \quad Y_k^i = (y_1, \ldots y_{i-1}, y_{i+1}, y_k).$$

Sometimes we shall write y_1 or the value of this component instead of Y_1 if $k = 1$, and (y_1, y_2) or their values instead of Y_2 if $k = 2$. We also shall use the notation $(Y_k, z) = (y_1, \ldots, y_k, z)$.

We shall characterize the process (29) by the following functions:

$$G(k_1, k_2, x, y_1, \ldots, y_{k_2}, t) =$$
$$= \mathbf{P}\left\{\eta_1(t) = k_1, \eta_2(t) = k_2, \sigma(t) < x, \xi_*^j(t) < y_j, j = \overline{1, k_2}\right\},$$
$$k_1 = \overline{0, n+m},\ k_2 = \overline{1, r}; \quad (30)$$
$$W(k_1, k_2, y_1, \ldots, y_{k_2}, t) =$$
$$= \mathbf{P}\left\{\eta_1(t) = k_1, \eta_2(t) = k_2, \xi_*^j(t) < y_j, j = \overline{1, k_2}\right\} =$$
$$= G(k_1, k_2, V, y_1, \ldots, y_{k_2}, t),\ k_1 = \overline{0, n+m},\ k_2 = \overline{1, r}; \quad (31)$$
$$P(k_1, k_2, t) = \mathbf{P}\left\{\eta_1(t) = k_1, \eta_2(t) = k_2\right\},$$
$$k_1 = \overline{0, n+m},\ k_2 = \overline{0, r}. \quad (32)$$

If $k_2 = \overline{1, r}$, then
$$P(k_1, k_2, t) = W(k_1, k_2, \infty, \ldots, \infty, t),$$
where $\infty_k = (\infty, \ldots, \infty)$ is a k-component vector.

The steady state conditions for the model under consideration are obviously always satisfied, if V is finite. So the next limits exist:

$$g(k_1, k_2, x, y_1, \ldots, y_{k_2}) = \lim_{t \to \infty} G(k_1, k_2, x, y_1, \ldots, y_{k_2}, t),$$
$$k_1 = \overline{0, n+m},\ k_2 = \overline{1, r}; \quad (33)$$
$$w(k_1, k_2, y_1, \ldots, y_{k_2}) = \lim_{t \to \infty} W(k_1, k_2, y_1, \ldots, y_{k_2}, t) =$$
$$= g(k_1, k_2, V, y_1, \ldots, y_{k_2}),\ k_1 = \overline{0, n+m},\ k_2 = \overline{1, r}; \quad (34)$$
$$p(k_1, k_2) = \lim_{t \to \infty} P(k_1, k_2, t),\ k_1 = \overline{0, n+m},\ k_2 = \overline{0, r}. \quad (35)$$

If $k_2 = \overline{1, r}$, then $p(k_1, k_2) = w(k_1, k_2, \infty, \ldots, \infty)$.

Let us introduce the following notation:
$$\kappa = \min(k, n), \quad \gamma = \min(k+1, n).$$
It can be easily shown by means of process (29) analysis, that the functions (30)–(32) satisfy the following partial differential equations:

$$\frac{\partial P(0,0,t)}{\partial t} = -[a_1 L_1(V) + a_2 L_2(V)] P(0,0,t) +$$
$$+\mu P(1,0,t) + \frac{\partial W(0,1,y,t)}{\partial y}\Big|_{y=0}; \quad (36)$$

$$\frac{\partial P(1,0,t)}{\partial t} = a_1 L_1(V) P(0,0,t) - a_1 \int_0^V G(1,0,V-x,t) dL_1(x) -$$

$$-\mu P(1,0,t) - -a_2 \int_0^V G(1,0,V-x,t) dL_2(x) +$$

$$+2\mu P(2,0,t) + \frac{\partial W(1,1,y,t)}{\partial y}\Big|_{y=0}; \qquad (37)$$

$$\frac{\partial P(k,0,t)}{\partial t} = a_1 \int_0^V G(k-1,0,V-x,t) dL_1(x) -$$

$$-(1-\delta_{k,n+m}) a_1 \int_0^V G(k,0,V-x,t) dL_1(x) - \kappa\mu P(k,0,t) -$$

$$-a_2 \int_0^V G(k,0,V-x,t) dL_2(x) + (1-\delta_{k,n+m}) \gamma\mu P(k+1,0,t) +$$

$$+\frac{\partial W(k,1,y,t)}{\partial y}\Big|_{y=0}, \; k = \overline{2, n+m}; \qquad (38)$$

$$\frac{\partial W(0,1,y,t)}{\partial t} - \frac{\partial W(0,1,y,t)}{\partial y} + \frac{\partial W(0,1,y,t)}{\partial y}\Big|_{y=0} =$$

$$= a_2 P(0,0,t) F(V,y) - a_1 \int_0^V G(0,1,V-x,y,t) dL_1(x) -$$

$$-a_2 \int_0^V G(0,1,V-x,y,t) dL_2(x) +$$

$$+\mu W(1,1,y,t) + \frac{\partial W(0,2,y,z,t)}{\partial z}\Big|_{z=0}; \qquad (39)$$

$$\frac{\partial W(0,k,Y_k,t)}{\partial t} - \frac{1}{k}\sum_{i=1}^{k}\left[\frac{\partial W(0,k,Y_k,t)}{\partial y_i} - \frac{\partial W(0,k,Y_k,t)}{\partial y_i}\Big|_{y_i=0}\right] =$$

$$= \frac{a_2}{k}\sum_{i=1}^{k} G(0,k-1,V-x,Y_k^i,t) d_x F(x,y_i) +$$

$$+\mu W(1,k,Y_k,t) - a_1 \int_0^V G(0,k,V-x,Y_k,t) dL_1(x) +$$

$$+(1-\delta_{k,r})\left[\frac{\partial W(0,k+1,(Y_k,z),t)}{\partial z}\Big|_{z=0} -\right.$$

$$\left. -a_2 \int_0^V G(0,k,V-x,Y_k,t) dL_2(x)\right], \; k = \overline{2, r}; \qquad (40)$$

$$\frac{\partial W(k_1,k_2,Y_{k_2},t)}{\partial t} - \frac{1}{k_2}\sum_{i=1}^{k_2}\left[\frac{\partial W(k_1,k_2,Y_{k_2},t)}{\partial y_i} - \right.$$

$$\left. - \frac{\partial W(k_1,k_2,Y_{k_2},t)}{\partial y_i}\Big|_{y_i=0}\right] = a_1\int_0^V G(k_1-1,k_2,V-x,Y_{k_2},t)dL_1(x) +$$

$$+\frac{a_2}{k_2}\sum_{i=1}^{k_2}\int_0^V G(k_1,k_2-1,V-x,Y_{k_2}^i,t)d_xF(x,y_i) +$$

$$+(1-\delta_{k_1,n+m})\left[\gamma\mu W(k_1+1,k_2,Y_{k_2},t) - \right.$$

$$\left. - a_1\int_0^V G(k_1,k_2,V-x,Y_{k_2},t)dL_1(x)\right] -$$

$$-\kappa\mu W(k_1,k_2,Y_{k_2},t) + (1-\delta_{k_2,r})\left[\frac{\partial W(k_1,k_2+1,(Y_{k_2},z),t)}{\partial z}\Big|_{z=0} - \right.$$

$$\left. - a_2\int_0^V G(k_1,k_2,V-x,Y_{k_2},t)dL_2(x)\right],\ k_1 = \overline{1,m+n},\ k_2 = \overline{1,r}. \quad (41)$$

It follows from equations (36)–(41), that the stationary functions (33)–(35) satisfy the following equations:

$$0 = -[a_1L_1(V) + a_2L_2(V)]p(0,0) + \mu p(1,0) + \frac{\partial w(0,1,y)}{\partial y}\Big|_{y=0}; \quad (42)$$

$$0 = a_1L_1(V)p(0,0) - a_1\int_0^V g(1,0,V-x)dL_1(x) - \mu p(1,0) -$$

$$-a_2\int_0^V g(1,0,V-x)dL_2(x) + 2\mu p(2,0) + \frac{\partial w(1,1,y)}{\partial y}\Big|_{y=0}; \quad (43)$$

$$0 = a_1\int_0^V g(k-1,0,V-x)dL_1(x) -$$

$$-(1-\delta_{k,n+m})a_1\int_0^V g(k,0,V-x)dL_1(x) -$$

$$-\kappa\mu p(k,0) - a_2\int_0^V g(k,0,V-x)dL_2(x) +$$

$$+(1-\delta_{k,n+m})\gamma\mu p(k+1,0) + \frac{\partial w(k,1,y)}{\partial y}\Big|_{y=0},\ k = \overline{2,n+m}; \quad (44)$$

$$-\frac{\partial w(0,1,y)}{\partial y}+\frac{\partial w(0,1,y)}{\partial y}\Big|_{y=0}=$$

$$=a_2p(0,0)F(V,y)-a_1\int_0^V g(0,1,V-x,y)dL_1(x)-$$

$$-a_2\int_0^V g(0,1,V-x,y)dL_2(x)+\mu w(1,1,y)+\frac{\partial w(0,2,y,z)}{\partial z}\Big|_{z=0}; \quad (45)$$

$$-\frac{1}{k}\sum_{i=1}^{k}\left[\frac{\partial w(0,k,Y_k)}{\partial y_i}-\frac{\partial w(0,k,Y_k)}{\partial y_i}\Big|_{y_i=0}\right]=$$

$$=\frac{a_2}{k}\sum_{i=1}^{k}g(0,k-1,V-x,Y_k^i)d_xF(x,y_i)+\mu w(1,k,y_1,\ldots,y_k)-$$

$$-a_1\int_0^V g(0,k,V-x,Y_k)dL_1(x)+$$

$$+(1-\delta_{k,r})\left[\frac{\partial w(0,k+1,(Y_k,z))}{\partial z}\Big|_{z=0}-\right.$$

$$\left.-a_2\int_0^V g(0,k,V-x,Y_k)dL_2(x)\right],\ k=\overline{2,r}; \quad (46)$$

$$-\frac{1}{k_2}\sum_{i=1}^{k_2}\left[\frac{\partial w(k_1,k_2,Y_{k_2})}{\partial y_i}-\frac{\partial w(k_1,k_2,Y_{k_2})}{\partial y_i}\Big|_{y_i=0}\right]=$$

$$=a_1\int_0^V g(k_1-1,k_2,V-x,Y_{k_2})dL_1(x)+$$

$$+\frac{a_2}{k_2}\sum_{i=1}^{k_2}\int_0^V g(k_1,k_2-1,V-x,Y_{k_2}^i)d_xF(x,y_i)+$$

$$+(1-\delta_{k_1,n+m})\left[\gamma\mu w(k_1+1,k_2,Y_{k_2})-\right.$$

$$\left.-a_1\int_0^V g(k_1,k_2,V-x,Y_{k_2})dL_1(x)\right]-$$

$$-\kappa\mu w(k_1,k_2,Y_{k_2})+(1-\delta_{k_2,r})\left[\frac{\partial w(k_1,k_2+1,(Y_{k_2},z))}{\partial z}\Big|_{z=0}-\right.$$

$$\left.-a_2\int_0^V g(k_1,k_2,V-x,Y_{k_2})dL_2(x)\right],\ k_1=\overline{1,m+n},\ k_2=\overline{1,r}. \quad (47)$$

We also have to take into account the following boundary condition

$$(1-\delta_{k_1,n+m})\gamma\mu w(k_1+1,k_2,Y_{k_2})+$$
$$+(1-\delta_{k_2,r})\frac{\partial w(k_1,k_2+1,(Y_{k_2},z))}{\partial z}\Big|_{z=0}=$$
$$=(1-\delta_{k_1,n+m})a_1\int_0^V g(k_1,k_2,V-x,Y_{k_2})dL_1(x)+$$
$$+(1-\delta_{k_2,r})a_2\int_0^V g(k_1,k_2,V-x,Y_{k_2})dL_2(x),$$
$$k_1=\overline{1,m+n},\ k_2=\overline{1,r}, \tag{48}$$

which follows from the fact that in the analyzed combination admitting to the system during some time interval of k customers probability is equal (in steady state) to probability, that k customers complete their service during this interval.

The normalization condition for the combination has the following form:

$$\sum_{k_1=0}^{n+m}\sum_{k_2=0}^{r}p(k_1,k_2)=1. \tag{49}$$

Let us introduce the following notation:

$$N(k)=\begin{cases}\dfrac{(n\rho_1)^k}{k!}, & k=\overline{0,n},\\ \dfrac{n^n\rho_1^k}{n!}, & k=\overline{n+1,n+m},\end{cases}$$

where $\rho_1=a_1/(n\mu)$.

We also introduce the following functions:

$$H(x,y)=\mathbf{P}\{\zeta_2<x,\xi_2\geq y\}=\int_{v=0}^{x}\int_{u=y}^{\infty}dF(v,u)=$$
$$=\mathbf{P}\{\zeta_2<x\}-\mathbf{P}\{\zeta_2<x,\xi_2<y\}=L_2(x)-F(x,y);$$
$$\Phi_y(x)=\int_0^y H(x,u)du=L_2(x)\int_0^y[1-B_2(u|\zeta_2<x)]dy,$$

where $B(u|\zeta_2<x)=\mathbf{P}\{\xi_2<u|\zeta_2<x\}$.

Taking into account the symmetry of the functions

$$g(k_1,k_2,x,y_1,\dots,y_{k_2}) \text{ and } w(k_1,k_2,y_1,\dots,y_{k_2})$$

with respect to the variables $y_1,\dots,y_{k_2}$ (because of random numbering of customers in the second system), it can be easy shown, that the functions defined by the relation

$$g(k_1,k_2,x,Y_{k_2}) = p(0,0)N(k_1){a_2}^{k_2}\left[L_1^{(k_1)} * \left(\mathop{*}_{i=1}^{k_2} \Phi_{y_i}\right)\right](x) \qquad (50)$$

satisfy equations (42)–(48). It follows from relation (50) that

$$w(k_1,k_2,Y_{k_2}) = p(0,0)N(k_1){a_2}^{k_2}\left[L_1^{(k_1)} * \left(\mathop{*}_{i=1}^{k_2} \Phi_{y_i}\right)\right](V) \qquad (51)$$

Denote as [9]

$$D(x) = \lim_{y\to\infty}\Phi_y(x) = L_2(x)\int_0^\infty [1 - B_2(u|\zeta_2 < x)]du =$$

$$= \int_{u=0}^{x}\int_{y=0}^{\infty} dF(u,y).$$

Then from (50), (51) we obtain

$$p(k_1,k_2) = w(k_1,k_2,\infty_{k_2}) = p(0,0)N(k_1){a_2}^{k_2}L_1^{(k_1)} * D^{(k_2)}(V).$$

The coefficient $p(0,0)$ can be determined from the normalization condition (49):

$$p(0,0) = \left[\sum_{k_1=0}^{n+m}\sum_{k_2=0}^{r} N(k_1){a_2}^{k_2}L_1^{(k_1)} * D^{(k_2)}(V)\right]^{-1}.$$

If we denote the probability that there are k_1 customers in the first queue as $p_1(k_1) = \sum\limits_{k_2=0}^{r} p(k_1,k_2)$, the loss probability for this queue have obviously the following form:

$$p_L^1 = 1 - (n\rho_1)^{-1}\sum_{k_1=1}^{n-1} k_1 p_1(k_1) - {\rho_1}^{-1}\left(1 - \sum_{k_1=1}^{n-1} p_1(k_1)\right).$$

We introduce the following notation:

$$w_2(k_2, Y_{k_2}) = \sum_{k_1=0}^{m+n} w(k_1, k_2, Y_{k_2}) =$$

$$= p(0,0) \sum_{k_1=0}^{m+n} N(k_1) a_2{}^{k_2} \left[L_1^{(k_1)} * \left(\underset{i=1}{\overset{k_2}{*}} \Phi_{y_i} \right) \right] (V). \qquad (52)$$

The loss probability for the second system can be obviously obtained from the next equilibrium equation:

$$a_2(1 - p_L^2) = \sum_{k_2=1}^{r} \frac{\partial w_2(k_2, (\infty_{k_2-1}, z))}{\partial z} \Big|_{z=0}. \qquad (53)$$

Then from (53), (54) and from the fact that $\frac{\partial \Phi_z(x)}{\partial z} \Big|_{z=0} = L_2(x)$ we finally obtain the loss probability for the second system in the following form:

$$p_L^2 = 1 - p(0,0) \sum_{k_2=0}^{r-1} \sum_{k_1=0}^{m+n} N(k_1) a_2{}^{k_2} L_1^{(k_1)} * D^{(k_2)} * L_2(V).$$

It is clear that we can always analyze other types of queueing systems with non-homogeneous customers connected via common memory space.

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