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W–irreducible Lattices*

Joanna Grygiel

Institute of Mathematics and Computer Science
Jan Długosz University of Częstochowa
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
e-mail: j.grygiel@ajd.czest.pl

Abstract
A finite lattice is W-irreducible if it cannot be split into two overlapping lattices, one of them being an ideal and the other a filter of the lattice. We give some characterization of finite W-irreducible lattices.

In 1974 Andrzej Wroński ([7]) introduced an operation $\oplus$ of sum of lattices, which we will call a Wroński's sum (or shortly, a W-sum) of lattices. Its definition is following: let $\mathcal{L}_0 = \langle L_0, \leq_0\rangle$ and $\mathcal{L}_1 = \langle L_1, \leq_1\rangle$ be lattices such that $L_0 \cap L_1$ is a filter in $\mathcal{L}_0$ and an ideal in $\mathcal{L}_1$ and the orderings $\leq_0$ and $\leq_1$ coincide on $L_0 \cap L_1$, then $\mathcal{L}_0 \oplus \mathcal{L}_1 = \langle L_0 \cup L_1, \leq\rangle$, where $\leq$ is the transitive closure of $\leq_0 \cup \leq_1$, is a W-sum of lattices $\mathcal{L}_0$ and $\mathcal{L}_1$.

It can be shown that the W-sum of lattices (if there exists) is a lattice, what is more, it preserves some important properties of lattices, like distributivity or modularity. However, it does not mean that all lattice identities are preserved, for example the Desargues identity is not, since the W-sum of two arguesian intervals can be non-arguesian (see [1]).

The W-sum of two finite lattices can be regarded as a special case of $\mathcal{K}$-gluing introduced by Christian Herrmann in [5].

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Let $(\mathcal{L}_x)_{x\in K}$ be a family of finite lattices and let the index set K be also a finite lattice. We call the family $(\mathcal{L}_x)_{x\in K}$ a $\mathcal{K}$-atlas with overlapping neighbours if the following conditions hold for every $x, y \in K$:

1. If $L_x \subseteq L_y$ then $x = y$.

2. If $x \prec y$ then $L_x \cap L_y \neq \emptyset$.

3. If $x \leq y$ and $L_x \cap L_y \neq \emptyset$ then the orders of $\mathcal{L}_x$ and $\mathcal{L}_y$ coincide on the intersection $L_x \cap L_y$ and the interval $L_x \cap L_y$ is at the same time a filter of $\mathcal{L}_x$ and an ideal of $\mathcal{L}_y$.

4. $L_x \cap L_y = L_{x\wedge y} \cap L_{x\vee y}$.

The structure $\mathcal{L} = \langle \bigcup\limits_{x\in K} L_x, \leq \rangle$, where $\leq$ is the transitive closure of the union of orders of the lattices $\mathcal{L}_x$ for $x \in K$, is called the sum of $\mathcal{K}$-atlas with overlapping neighbours (or simply a $\mathcal{K}$-gluing of the family $(\mathcal{L}_x)_{x\in K}$).

Thus, it is clear that if finite lattices $\mathcal{L}_0$ and $\mathcal{L}_1$ fulfill Wroński's conditions and no one is contained in the another, then their W-sum is the gluing of the $\mathcal{K}$-atlas $(\mathcal{L}_0, \mathcal{L}_1)$, where $\mathcal{K}$ is a two-element boolean lattice. Let us observe, however, that since we can repeat the Wroński's sum of two lattices finitely many times, it is reasonable to regard it, in general, as an iteration of the $\mathcal{K}$-gluing with a two-element skeleton $\mathcal{K}$. For some reasons, it is also important to skip the first condition of the $\mathcal{K}$-atlas with overlapping neighbours. In can be proved that the W-sum and $\mathcal{K}$-gluing do not coincide. We discussed this problem carefully in [3].

A lattice $\mathcal{L}$ is H-irreducible if there is no nontrivial lattice $\mathcal{K}$ and a $\mathcal{K}$-atlas $(\mathcal{L}_x)_{x\in K}$ such that $\mathcal{L}$ is the $\mathcal{K}$-gluing of the atlas. We gave some characterization of H-irreducible lattices in [4].

It is said that a lattice $\mathcal{L}$ is W-irreducible iff there are no proper sublattices $\mathcal{L}_0$ and $\mathcal{L}_1$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$.

It is obvious that every H-irreducible lattice is W-irreducible but not conversely. Here is a natural characterization of W-irreducible finite lattices:

Theorem 1. *For every finite lattice $\mathcal{L}$ the following conditions are equivalent:*

1. *$\mathcal{L}$ is W-irreducible;*
2. *$\bigvee N(a) \vee a = 1_{\mathcal{L}}$ for every $a \in L \setminus \{0_{\mathcal{L}}\}$;*
3. *$\bigwedge N(a) \wedge a = 0_{\mathcal{L}}$ for every $a \in L \setminus \{1_{\mathcal{L}}\}$,*

where $N(a)$ is the set of all elements of $\mathcal{L}$ which are not comparable with a.

Proof.

Let us suppose that $x = \bigvee N(a) \vee a < 1_{\mathcal{L}}$ for some $a > 0_{\mathcal{L}}$.

Let $L_0 = [0_{\mathcal{L}}, x]$, $L_1 = [a, 1_{\mathcal{L}}]$. Since $x \geq a$, we get $L_0 \cap L_1 = [a, x]$, which is a filter in $\mathcal{L}_0$ and an ideal in $\mathcal{L}_1$. Thus $\mathcal{L}_0 \oplus \mathcal{L}_1$ is a sublattice of the lattice $\mathcal{L}$.

On the other hand, let $y \in L$. If $y \geq a$ then $y \in L_1$. If $y \leq a$ then $y \leq x$ and hence $y \in L_0$. In the case when a and y are incomparable we have $y \in L_0$. It means that $\mathcal{L}_0 \oplus \mathcal{L}_1 = \mathcal{L}$ and both $\mathcal{L}_0$ and $\mathcal{L}_1$ are proper sublattices of $\mathcal{L}$, which contradicts the assumption.

Now, let us suppose that $\mathcal{L}_0 \oplus \mathcal{L}_1 = \mathcal{L}$, where $\mathcal{L}_0$ and $\mathcal{L}_1$ are proper sublattices of $\mathcal{L}$. Then $L_0 = [0_{\mathcal{L}}, x]$ and $L_1 = [y, 1_{\mathcal{L}}]$, for some $y \leq x < 1_{\mathcal{L}}$. If $N(y) = \emptyset$ then $\bigvee N(y) \vee y = y < 1_{\mathcal{L}}$. Let $a \in N(y)$. Thus $a \in L_0$ and hence $a \leq x$. It yields $\bigvee N(y) \vee y \leq x < 1_{\mathcal{L}}$.

Thus, we proved the equivalence of 1. and 2. The proof of the equivalence of 1. and 3. is analogous. •

The above Theorem describes when a finite lattice can be split into two overlapping parts, one of them being an ideal and the other a filter of the lattice. It is easy to observe that it cannot be done to any finite boolean lattice, which means that finite boolean lattices are W-irreducible. What is more

Corollary 2. A finite distributive lattice is W-irreducible iff it is a boolean lattice.

Proof

Let us suppose that $\mathcal{B}$ is a finite W-irreducible distributive lattice. Then, by Theorem 1, for every nonzero $a \in B$ holds $\bigvee N(a) \vee a = 1_{\mathcal{B}}$. Let $a \in At(\mathcal{B})$. Then $a \wedge \bigvee N(a) = 0_{\mathcal{B}}$. Thus, every atom of $\mathcal{B}$ has a complement and therefore there is a complement b of $\bigvee At(\mathcal{B})$. Since $b \wedge a = 0_{\mathcal{B}}$ for every $a \in At(\mathcal{B})$ then $b = 0_{\mathcal{B}}$, which yields

$$b' = \bigvee At(\mathcal{B}) = 1_{\mathcal{B}}.$$

We conclude that $\mathcal{B}$ is atomistic and hence $\mathcal{B}$ is a boolean lattice. •

The notion of W-irreducibility can also be characterized in a different way.

Theorem 3. *A finite lattice $\mathcal{K}$ is W-irreducible iff for every finite lattices $\mathcal{L}_0$ and $\mathcal{L}_1$ such that there exists $\mathcal{L}_0 \oplus \mathcal{L}_1$*

$$(*) \;\; K \text{ is an interval of } \mathcal{L}_0 \oplus \mathcal{L}_1 \text{ iff}$$
$$(K \text{ is an interval of } L_0 \text{ or } K \text{ is an interval of } L_1).$$

Proof.

If $\mathcal{K}$ is not W-irreducible then there are proper intervals $\mathcal{L}_0$ and $\mathcal{L}_1$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{L}_0 \oplus \mathcal{L}_1$. Thus $K \subseteq L_0 \cup L_1$ but $K \not\subseteq L_0$ and $K \not\subseteq L_1$, which means that (*) does not hold.

Let us suppose now that there are finite lattices $\mathcal{L}_0$ and $\mathcal{L}_1$ such that $\mathcal{L}_0 \oplus \mathcal{L}_1$ exists and K is an interval of $\mathcal{L}_0 \oplus \mathcal{L}_1$ but $K \not\subseteq L_0$ and $K \not\subseteq L_1$. We shall prove that $\mathcal{K}$ is not W-irreducible. In particular, we are going to show that

$$\mathcal{K} = (\mathcal{K} \cap \mathcal{L}_0) \oplus (\mathcal{K} \cap \mathcal{L}_1)$$

and $\mathcal{K} \cap \mathcal{L}_0$ and $\mathcal{K} \cap \mathcal{L}_1$ are proper sublattices of $\mathcal{K}$.

Let us denote, for simplicity, $L_0 = [0_0, 1_0]$ and $L_1 = [0_1, 1_1]$. Then, by the assumptions

$$0_0 \leq 0_{\mathcal{K}} \leq 1_1 \text{ and } 0_1 \leq 1_0.$$

Thus

$$K \cap L_0 = [0_{\mathcal{K}}, 1_{\mathcal{K}} \wedge 1_0] \neq \emptyset,$$

$$K \cap L_1 = [0_{\mathcal{K}} \vee 0_1, 1_{\mathcal{K}}] \neq \emptyset.$$

Then $0_{\mathcal{K}} \leq 1_{\mathcal{K}} \wedge 1_0$ and $0_{\mathcal{K}} \vee 0_1 \leq 1_{\mathcal{K}}$, so $0_{\mathcal{K}} \vee 0_1 \leq 1_{\mathcal{K}} \wedge 1_0$, which proves that

$$K \cap L_0 \cap L_1 = [0_{\mathcal{K}} \vee 0_1, 1_{\mathcal{K}} \wedge 1_0] \neq \emptyset.$$

Since $K \cap L_0 \cap L_1$ is a filter of $\mathcal{K} \cap \mathcal{L}_0$ and an ideal of $\mathcal{K} \cap \mathcal{L}_1$ then there exists $(\mathcal{K} \cap \mathcal{L}_0) \oplus (\mathcal{K} \cap \mathcal{L}_1)$. Moreover, as $1_{\mathcal{K}} \wedge 1_0 < 1_{\mathcal{K}}$ and $0_{\mathcal{K}} < 0_{\mathcal{K}} \vee 0_1$ then $\mathcal{K} \cap L_0$ and $\mathcal{K} \cap \mathcal{L}_1$ are proper sublattices of $\mathcal{K}$ and

$$\mathcal{K} = (\mathcal{K} \cap \mathcal{L}_0) \oplus (\mathcal{K} \cap \mathcal{L}_1). \quad \bullet$$

As a consequence of Theorem 3, we get

Corollary 4. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributive lattices such that there exists $\mathcal{D}_1 \oplus \mathcal{D}_2$ and let $\mathcal{B}$ be a boolean lattice. If $\mathcal{B}$ is an interval of $\mathcal{D}_1 \oplus \mathcal{D}_2$ then $\mathcal{B}$ is an interval of D_1 or $\mathcal{B}$ is an interval of D_2.

Let us notice that every finite lattice can be represented as a W-sum of its W-irreducible intervals. This observation together with Corollary 4 yields the following theorem, proved by Kotas, Wojtylak:

Theorem 5. (see [6]) *Every finite distributive lattice can be represented as a W-sum of its Boolean intervals.*

Each representation of a finite lattice $\mathcal{K}$ as a W-sum of its W-irreducible intervals is called a W-representation of $\mathcal{K}$. The set of all W-representations of a given lattice was discussed in [2].

References

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