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Sublinear Functionals and Weak*-compactness

Roman Ger
Institute of Mathematics and Computer Science,
Jan Długosz University of Częstochowa,
ul. Armii Krajowej 13/15, 42-201 Częstochowa, Poland
e-mail r.ger@ajd.czest.pl

Abstract
Let X be a real locally convex linear topological space. A functional $f : X \longrightarrow \mathbb{R}$ is called *sublinear* provided that f is subadditive and $f(nx) = nf(x), x \in X, n \in \mathbb{N}$. We establish a one-to-one correspondence between the collection of all sublinear functional satisfying some mild regularity conditions and the family of all nonempty convex and weakly*-compact subsets of the dual space X^*.

1. Introduction

A real functional φ on a real linear space X is termed *sublinear* provided that φ is *subadditive*, i.e.

$$\varphi(x+y) \leq \varphi(x) + \varphi(y)\,, \quad x, y \in X\,,$$

and

$$\varphi(nx) = n\varphi(x)\,, \quad x \in X\,,\, n \in \mathbb{N}\,.$$

It is known (folklore) that the latter requirement my equivalently be replaced by an apparently weaker condition

$$\varphi(2x) = 2\varphi(x)\,, \quad x \in X.$$

If D is a nonempty convex subdomain of X then a functional $\varphi : D \longrightarrow \mathbb{R}$ is called *Jensen-convex* provided that

$$\varphi\left(\frac{x+y}{2}\right) \le \frac{\varphi(x)+\varphi(y)}{2}$$

for all $x, y \in D$.

Assuming that D is a nonempty open and convex subdomain of a real linear topological space X one encounters numerous results showing that whenever a Jensen-convex functional on D is supposed to satisfy some additional regularity condition then necessarily it has to be continuous and hence also convex, i.e. to satisfy the inequality

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all $x, y \in D$ and all $\lambda \in [0, 1]$. As an instance, let us quote some of the conditions of that kind (see e.g. M.Kuczma [7, Chapter IX, §3, Theorem 3]):

(a) φ admits a measurable majorant on a set of positive measure (in the case where $X = \mathbb{R}$);

(b) φ is upper bounded on a second category Baire subset of X;

(c) φ is upper bounded on a set $T \subset \mathbb{R}^n$ such that its Q-convex hull $\mathrm{Q}(T)$ is of positive inner Lebesgue measure;

(d) φ is upper bounded on a set $T \subset X$ such that $\mathrm{Q}(T)$ contains a second category Baire set.

In 1975 E. Berz [1, Corollary 1.7] has proved, among others, that every Lebesgue measurable sublinear functional $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is necessarily of the form

$$(\alpha\beta) \qquad \varphi(x) = \begin{cases} \alpha x & \text{for} \quad x < 0 \\ \beta x & \text{for} \quad x \ge 0\,, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha \le \beta$.

With the aid of an entirely different and quite elementary method B. Kocl/ega and the present author have proved in [5] the following generalization of this result.

Theorem A. *A sublinear functional* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is of the form* $(\alpha\beta)$ *where* $\alpha, \beta \in \mathbb{R}$, $\alpha \le \beta$, *if and only if*

$$(\mathrm{J}) \quad \left\{ \begin{array}{l} \varphi \;\; satisfies \;\; any \;\; regularity \;\; condition \;\; that \;\; forces \\ Jensen-convex function \;\; to \;\; be \;\; continuous. \end{array} \right.$$

Observe that functions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ of the form $(\alpha\beta)$ may alternatively be expressed in the form

$$(\gamma) \qquad \varphi(x) = \sup\{\gamma x : \ \gamma \in [\alpha, \beta]\}\ , \quad x \in X\ .$$

Therefore, in higher dimensional spaces one could expect a kind of interplay between mildly regular sublinear functionals and compact convex subsets of the space in question. In what follows we shall show that that is really the case.

The results spoken of in this paper were presented by the author at the 36-th International Symposium on Functional Equations (Brno, Czech Republic, 1998); see [4].

2. Main results

With no regularity whatsoever one can hardly imagine any reasonable description of sublinear functionals even on the real line. Indeed, any discontinuos additive selfmapping of $\mathbb{R}$ may serve as an example of nonmeasurable sublinear function that fails to be of the form $(\alpha\beta)$ and hence also (γ). It turns out that, likewise in Theorem A, condition (J) is necessary and sufficient for getting the description desired.

Theorem 1. *Let* X *be a real locally convex Hausdorff linear topological space and let* $\varphi : X \longrightarrow \mathbb{R}$ *be a sublinear functional satisfying the* (J) *condition. Then, there exists exactly one nonempty convex and weakly***-compact set* $A \subset X^*$ *such that*

$$(A) \qquad \varphi(x) = \sup\{x^*(x) : \ x^* \in A\}\ , \quad x \in X\ .$$

Conversely, if X *is a Baire space (i.e. is of the second Baire category) for every nonempty convex and weakly***-compact set* $A \subset X^*$

formula (A) defines a real sublinear functional on X that is Jensen convex and continuous.

In other words, in every locally compact Baire space there is a one-to-one correspondence between the collections of all sublinear functionals satisfying the (J) *condition and all nonempty convex and weakly*-compact subsets of this space.*

Proof. Let $\varphi : X \longrightarrow \mathbb{R}$ be a sublinear functional enjoying the (J) property. It is easy to see that any sublinear functional is necessarily $\mathbb{Q} \cap (0, \infty)$-homogeneous. Since, for every $x, y \in X$ one has

$$\varphi\left(\frac{x+y}{2}\right) = \frac{1}{2}\,\varphi(x+y) \leq \frac{\varphi(x)+\varphi(y)}{2}\,,$$

we see that φ is Jensen convex and hence continuous. By a theorem of E. Berz [1, Theorem 1.4 jointly with Remark 2.1] we get the formula

$$\varphi(x) = \sup\{f(x) :\ f : X \longrightarrow \mathbb{R} \text{ is additive and } f \leq \varphi\}, \quad x \in X.$$

Due to the continuity of φ all the additive functionals f occurring under the sup sign above have to be continuous as well and then automatically linear (recall that the space X is supposed to be a real one). Consequently, one has

$$\varphi(x) = \sup\{x^*(x) :\ x^* \in A\}, \quad x \in X,$$

where

$$A := \{x^*(x) \in X^* :\ x^*(x) \leq \varphi(x),\ x \in X\}.$$

Clearly, the set A is nonempty and convex and it is not hard to check (using the net technique, for instance, and bearing in mind that the weak*-convergence coincides with the pointwise convergence) that A is also weakly*-closed.

Now, because of the continuity of the functional φ at zero and the fact that φ vanishes at zero, one may find a convex and balanced neighbourhood U of the point $0 \in X$ such that $|\varphi(x)| \leq 1$ for all $x \in U$. Then the polar set

$$U^0 := \{x^* \in X^* :\ \sup_{x \in U} |x^*(x)| \leq 1\}$$

is weakly*-compact (a generalization of the celebrated Banach-Alaoglu theorem, see e.g. K. Yosida's monograph [8, p. 137]). Note that since

U is balanced, for every functional $x^* \in A$ and every $x \in U$ one has $|x^*(x)| \leq |\varphi(x)| \leq 1$ showing that the weakly*-closed set A is contained in the weakly*-compact set U^0. Thus A is weakly*-compact as well.

To prove the uniqueness of the representation (A), assume that for some nonempty convex and weakly*-compact set $B \subset X^*$ we have also

$$\varphi(x) = \sup\{x^*(x) : x^* \in B\}\ , \quad x \in X.$$

In such a case, for every $x^* \in B$ and every $x \in X$ we obtain an inequality $x^*(x) \leq \varphi(x)$, stating that $B \subset A$. Assume, for the indirect proof, that $B \neq A$. Then there exists a member x_0^* of A that fails to fall into B, i.e. $B \cap \{x_0^*\} = \emptyset$. Consequently, these two nonempty convex and weakly*-compact sets B and $\{x_0^*\}$ can be strictly separated in the sense that there exists a weakly*-continuous functional x_0^{**} on X^* such that

$$\sup_{x^* \in B} x_0^{**}(x^*) < x_0^{**}(x_0^*).$$

It is well known that x_0^{**} being weakly*-continuous has to have the form

$$x_0^{**}(x^*) = x^*(x_0), \quad x^* \in X^*,$$

for some $x_0 \in X$. Therefore

$$\begin{aligned} \varphi(x_0) &= \sup\{x^*(x_0) : x^* \in B\} = \sup\{x_0^{**}(x^*) : x^* \in B\} \\ &< x_0^{**}(x_0^*) = x_0^*(x_0) \leq \sup\{x^*(x_0) : x^* \in A\} = \varphi(x_0), \end{aligned}$$

a contradiction, which completes the proof of necessity.

Conversely, if a functional $\varphi : X \longrightarrow \mathbb{R}$ is of the form (A) with A being a nonempty convex and weakly*-compact subset of X^*, then it is obviously convex (in particular, Jensen-convex). Moreover, φ is lower-semicontinuous as a pointwise supremum of a family of continuous real functions. In particular, φ enjoys the (J) property because any lower-semicontinuous Jensen-convex functional on a Baire space is necessarily continuous (see e.g. Z. Kominek [6, Theorem 3.1]). Thus the proof has been completed.

In the case where the underlying space is a Banach one we have the following

Theorem 2. *Let* $(X, \|\cdot\|)$ *be a real Banach space and let* $\varphi : X \longrightarrow \mathbb{R}$ *be a sublinear functional satisfying the* (J) *condition.*

Then, there exists a nonempty closed convex and bounded set $A \subset X^*$ *such that the equality* $(*)$ *holds true. Such a set* A *is unique provided that the space* $(X, \|\cdot\|)$ *is reflexive.*

Conversely, for every nonempty bounded set $B \subset X^*$ *the formula*

$$(B) \qquad \varphi(x) = \sup\{x^*(x) : x^* \in B\}, \quad x \in X.$$

defines a real Lipschitzian sublinear functional on X.

Proof. Let $\varphi : X \longrightarrow \mathbb{R}$ be a sublinear functional enjoying the (J) property. We proceed like in the proof of Theorem 1 getting a representation $(*)$ with

$$A := \{x^* \in X^* : x^*(x) \le \varphi(x),\ x \in X\}$$

being nonempty and convex. To see that A is also closed it suffices to observe that the convergence of a sequence of elements of X^* in the strong topology implies the pointwise convergence of that sequence.

To show that A is bounded observe that φ being continuous at zero forces the existence of a positive δ such that $|\varphi(x)| \le 1$ for all x from a closed ball $\overline{B}(0, \delta)$ centered at zero and having radius δ. Because of the symmetry of that ball with respect to zero we infer that for every member x^* of A one has $|x^*(x)| \le 1$ whenever $x \in \overline{B}(0, \delta)$, which immediately implies that $\|x^*\| \le 1//\delta$; therefore $A \subset \overline{B}(0, 1//\delta)$.

The uniqueness of A may be derived along the same lines as in the proof of Theorem 1 with the only exception that now the representation

$$x_0^{**}(x^*) = x^*(x_0), \quad x^* \in X^*,$$

results from the reflexivity of X.

Conversely, if a functional $\varphi : X \longrightarrow \mathbb{R}$ is given by the formula (B) with B being a nonempty bounded subset of X^*, then having

$$\|x^*\| \le \varrho \quad \text{for all} \quad x^* \in B,$$

we obtain the inequalities

$$\varphi(x) \le \sup\{|x^*(x)| : x^* \in B\} \le \sup\{\|x^*\| \cdot \|x\| : x^* \in B\} \le \varrho\,\|x\|,$$

valid for all $x \in X$. Since, plainly, φ is sublinear we conclude that

$$\varphi(x) - \varphi(y) \le \varphi(x - y) \le \varrho\,\|x - y\|,$$

for all $x, y \in X$. Interchanging the roles of x and y we get finally

$$|\varphi(x) - \varphi(y)| \leq \varrho \,\|x - y\|, \; x, y \in X.$$

Thus φ is Lipschitzian, as claimed. This ends the proof.

Corollary 1. *Let $(X, (\cdot|\cdot))$ be a real Hilbert space and let $\varphi : X \longrightarrow \mathbb{R}$ be a sublinear functional satisfying the* (J) *condition. Then, there exists a unique nonempty convex and weakly compact set $A \subset X$ such that*

$$\varphi(x) = \sup\{(a|x) : \, a \in A\}\,, \quad x \in X.$$

Conversely, for every nonempty convex and weakly compact set $A \subset X$ the foregoing formula establishes a real Lipschitzian sublinear functional on X.

In other words, in every real Hilbert space there is a one-to-one correspondence between the collections of all sublinear functionals satisfying the (J) *condition and all nonempty convex and weakly compact subsets of this space.*

Proof. Any Hilbert space is reflexive and selfconjugate. Therefore, since closed convex sets are also weakly closed and in reflexive spaces the sets that are bounded and weakly closed are weakly compact, it remains to apply Theorem 2 and Riesz representation theorem for continuous linear functionals.

3. Christensen measurable sublinear functionals

Christensen measurability yields a generalization of the classical Haar measurability in locally compact groups onto the case of Polish topological groups. For the details the reader is referred to J.P.R. Christensen's monograph [2].

Theorem 3. *Let $(X, \|\cdot\|)$ be a real separable Banach space and let $\varphi : X \longrightarrow \mathbb{R}$ be a Christensen measurable sublinear functional. Then, there exists exactly one nonempty closed convex and bounded set $A \subset X^*$ such that the equality (A) holds true.*

Conversely, for every nonempty closed convex and bounded set $B \subset X^*$ the formula (B) defines a Christensen measurable sublinear functional on X.

Proof. Let $\varphi : X \longrightarrow \mathbb{R}$ be a Christensen measurable sublinear functional. Clearly, φ is Jensen-convex. P. Fischer and Z. Słodkowski [3] have proved, among others, that each Jensen-convex and Christensen measurable functional on a Polish linear space is automatically continuous. Thus φ enjoys the (J) property and it remains to apply Theorem 2.

The latter assertion results also from Theorem 2 and the fact that any Lipschitzian and *a fortiori* continuous function on a Polish space is Christensen measurable.

Corollary 2. *Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lebesgue measurable sublinear functional. Then, there exists exactly one nonempty compact convex set $A \subset \mathbb{R}^n$ such that*

$$\varphi(x) = \sup\{(a|x) : a \in A\}, \quad x \in X.$$

Conversely, for every nonempty compact convex set $A \subset \mathbb{R}^n$ the foregoing formula establishes a real Lipschitzian (and hence Lebesgue measurable) sublinear functional on $\mathbb{R}^n$.

Remark. For $n = 1$ Corollary 2 reduces to E. Berz's theorem spoken of in the Introduction.

4. Concluding remarks

We terminate this paper with a few observations based on the results presented above.

(a) Taking A to be the closed unit ball in the dual space X^* of a normed real linear space $(X, \|\cdot\|)$ we deal with a weakly*-compact set in X^* (Banach-Alaoglu theorem). The corresponding sublinear functional (A) yields then nothing else but the norm in X.

(b) Given a real Hausdorff locally compact space X, does any nonempty *symmetric* (with respect to 0), convex and weakly*-compact set $A \subset X^*$ produce a norm in X? In general, it does not.

However, it produces a seminorm in X. Actually, if a set $A \subset X^*$ enjoys all these properties, then formula (A) defines an even sublinear functional on X and, according to E. Berz's result [1, Corollary 1.8 jointly with Remark 2.1], there exist a real normed linear space $(Y, \|\cdot\|)$ and an additive map $L : X \longrightarrow Y$ such that

$$\varphi(x) := \sup\{x^*(x) : x \in A\} = \|L(x)\|,\ x \in X.$$

Obviously, because of the continuity of φ, that additive map L constitutes a continuous linear operator and the functional φ itself yields a seminorm in X.

Conversely, once φ is a seminorm on X, then there exists exactly one convex symmetric with respect to zero and weakly*-compact set $A \subset X^*$ such that equality (A) holds true; the existence of A results directly from our Theorem 1 whereas the symmetry follows from the evenness of a seminorm and the uniqueness stated in Theorem 1.

(c) Taking two distinct members a^*, b^* of the dual space X^* and considering a segment $A := \{\lambda a^* + (1-\lambda) b^* : \lambda \in [0,1]\}$ we get a weakly*-compact set generating (via (A)) a continuous sublinear functional on X

$$\varphi(x) = \begin{cases} b^*(x) & \text{whenever } a^*(x) \le b^*(x) \\ a^*(x) & \text{otherwise} \end{cases} \tag{1}$$

In the case where A is symmetric (i.e. $b^* = -a^*$) we arrive at the formula

$$\varphi(x) = |a^*(x)|,\ x \in X,$$

which becomes a special case of the sublinear functional φ derived in 2. in an entirely different way.

References

[1] E. Berz. Sublinear functions on $\mathbb{R}$. *Aequationes Math.*, **12**, pp. 200–206, 1975.

[2] J.P.R. Christensen. *Topology and Borel structure.* North Holland, Amsterdam & American Elsevier, New York, 1984.

[3] P. Fischer, Z. Słodkowski. Christensen zero sets and measurable convex functions. *Proc. Amer. Math. Soc.*, **79**, pp. 449-453, 1980.

[4] R. Ger. Sublinear functionals and weak∗-compactness. *Talk presented at the 36-th Int. Symp. on Functional Equations, May 24 - 30, 1998, Brno (Czech Republic)*; Report of Meeting: *Aequationes Math.*, **56**, p. 289, 1998.

[5] R. Ger, B. Koclęga. Isometries and a generalized Cauchy equation. *Aequationes Math.*, **60**, pp. 72–79, 2000.

[6] Z. Kominek. Convex functions in linear spaces. *Prace Naukowe Uniwersytetu Śląskiego w Katowicach*, **nr 1087**, Katowice, 1989.

[7] M. Kuczma. *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN & Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[8] K. Yosida. *Functional Analysis*, Sixth Edition, Springer-Verlag, Berlin-Heidelberg-New York, 1980.