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On Associative Rational Functions

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Abstract
We deal with the following problem: which rational functions of two variables are associative? We shall determine all of them provided that at least one of the coefficients in question vanishes.

1. Introduction

The term associativity is usually used in connection with a map F defined on the whole of the product $A \times A$ of a certain nonempty set A with values in A. That operation is called associative if for every $x, y, z \in A$ the following equation is satisfied:

$$F(x, F(y,z)) = F(F(x,y), z), \qquad x, y, z \in A \tag{E1}$$

Equation (E1) is then termed as the associativity equation.

For example, let us consider a map $F : I \times I \longrightarrow I$, where $I = (-1, 1)$, defined by the formula

$$F(x,y) = \frac{x+y}{1+xy}.$$

It is an operation in set I and it is associative which can easily be checked by a direct calculation.

It turns out that each map $F : I \times I \longrightarrow I$ in the form

$$F(x,y) = f^{-1}(f(x) + f(y)), \quad x, y \in I$$

is associative for any bijection $f : I \longrightarrow J$, where $I, J \subset \mathbb{R}$ are intervals such that $J + J \subset J$.

Moreover, in the class of continuous and bilaterally cancellative maps, these are the only associative ones. Namely the following theorem, proved by J. Aczel [1], (see also R. Craigen, Z. Páles [3] and others) is true:

Theorem CP. *Let I be a nontrivial real interval and let $F : I \times I \longrightarrow I$ be a continuous bilaterally cancellative associative operation. Then there exist a continuous bijection $f : I \longrightarrow J$ such that*

$$F(x,y) = f^{-1}(f(x) + f(y)), \quad x, y \in I, \tag{1}$$

where J is a (necessarily unbounded) real interval.

We notice that the above mentioned rational map is of the form (1), where $f : (-1, 1) \longrightarrow \mathbb{R}$, is defined by

$$f(x) := \mathrm{arctanh} x = \frac{1}{2} \log \frac{1+x}{1-x}, \quad x \in (-1, 1)$$

(clearly f is a bijection of $(-1, 1)$ onto $\mathbb{R}$ and

$$f^{-1}(x) = \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}, \quad x \in \mathbb{R}).$$

In practice, one is frequently faced to a situation where the map F considered is not necessarily defined on the entire „ rectangle" $A \times A$ but only for some pairs $(x, y) \in D \subset A \times A$. Nevertheless, one may still search for solutions $F : D \longrightarrow A$ of the *conditional* associativity equation

$$F(x, F(y, z)) = F(F(x, y), z) \tag{E2}$$

assumed to be satisfied for all triples $(x, y, z) \in A \times A \times A$ such that $(x, y), (y, z), (x, F(y, z))$ and $(F(x, y), z)$ all belong to D. And even in the case where $D = A \times A$ the values of F need not fall into A, therefore, we still have to deal with a conditional associativity. For example, taking a map $F : (-1,1) \times (-1,1) \longrightarrow \mathbb{R}$, given by the formula

$$F(x,y) = \frac{x+y}{1-xy}, \quad x,y \in (-1,1),$$

we have e.g. $F(\frac{1}{2},\frac{1}{2}), F(\frac{1}{3},\frac{1}{2})$ off $(-1,1)$. Nevertheless, F is conditionally associative which may easily be proved.

It turns out that a map of this type can be described analogously to (1), as proven by Gy. Maksa [4]:

Theorem M. *Let I be an open interval, $e \in I$ and let $F : I \times I \longrightarrow \mathbb{R}$ be continuous and strictly increasing in each variable. Suppose that*

$$F(x, F(y,z)) = F(F(x,y),z), \qquad x,y,z,F(x,y),F(y,z) \in I$$

and

$$F(x,e) = F(e,x) = x, \quad x \in I.$$

Then there exist $a,b \in I$ such that $a < e < b$ and there exists a continuous and strictly increasing function $f : [a,b] \longrightarrow [-1,1]$ such that

$$F(x,y) = f^{-1}(f(x)+f(y)), \quad x,y,F(x,y) \in [a,b] \tag{2}$$

and

$$f(a) = -1, \quad f(e) = 0, \quad f(b) = 1.$$

We notice that a map defined above is of the form (2), where $f : [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}] \longrightarrow [-1,1]$, is defined by

$$f(x) := \frac{3}{\pi}\text{arctanh}x, \quad x \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}],$$

which is easy to prove.

This paper concerns maps which belong to the field $\mathbb{R}[x,y]$ of quotients of the integral domain of all polynomials of two variables x,y, i.e. rational functions of these two variables. The domain considered is supposed to be of the form $\mathbb{R}^2 \setminus M$, where $M \subset \mathbb{R}^2$ is a set of planar Lebesgue measure zero (usually a curve). We will consider a family of maps of that kind defining them as operations. Rational function $F \in \mathbb{R}[x,y]$, which is of the form

$$F(x,y) = \frac{aW(x,y)}{bW(x,y)},$$

where W is polynomial of two variables x, y and $a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}$ ($M = \{(x,y) \in \mathbb{R}^2 : \quad W(x,y) = 0\}$) we will consider as constant $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Definition. Operation $F : \mathbb{R}^2 \setminus M \longrightarrow \mathbb{R}$, where $M \subset \mathbb{R}^2$ is a given set, is called *associative* iff F satisfies equation

$$F(x, F(y,z)) = F(F(x,y), z) \tag{E}$$

for all $(x,y,z) \in \mathbb{R}^3$ such that $(x,y), (y,z), (x, F(y,z)), (F(x,y), z) \notin M$.

It turns out that in order to be associative a rational function $F \in \mathbb{R}[x,y]$ has to be of special form, which is presented by the following theorem, proved by A. Chéritat [2]:

Theorem C. *If the rational function $F \in \mathbb{R}[x,y]$ is associative, then there exist $a, b, c, d, e, f, g, h \in \mathbb{R}$, $e^2 + f^2 + g^2 + h^2 > 0$, such that*

$$F(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h}. \tag{3}$$

The theorem gives only a necessary condition for a rational function to be associative. Not all operations of this form are associative, e.g. the following ones (on naturals domains)

$$F(x,y) = \frac{axy}{exy + h}, \qquad a, e, h \in \mathbb{R} \setminus \{0\},$$

$$F(x,y) = \frac{bx}{exy + h}, \qquad b, e, h \in \mathbb{R} \setminus \{0\},$$

$$F(x,y) = \frac{axy + bx}{exy + h}, \qquad a, b, e, h \in \mathbb{R} \setminus \{0\},$$

$$F(x,y)=\frac{bx}{fx+gy}, \qquad b,f,g\in \mathbb{R}\setminus\{0\},$$

$$F(x,y)=\frac{axy+bx}{fx+gy}, \qquad a,b,f,g\in \mathbb{R}\setminus\{0\},$$

$$F(x,y)=\frac{axy+bx}{exy+fx+gy}, \qquad a,b,e,f,g\in \mathbb{R}\setminus\{0\},$$

$$F(x,y)=\frac{axy+d}{exy+fx+gy}, \qquad a,d,e,f,g\in \mathbb{R}\setminus\{0\},$$

fail to be associative.

A sufficient condition allowing to choose associative operations from among those of the form (3) has not been formulated yet.

We find associative rational functions which belong to class functions of form (3), where at least one of the coefficients a,b,c,d,e,f,g,h is equal zero.

2. Main result

In what follows, from among rational functions of the form

$$F(x,y)=\frac{axy+bx+cy+d}{exy+fx+gy+h},$$

where $a,b,c,d,e,f,g,h\in\mathbb{R}$, $e^2+f^2+g^2+h^2>0$ we will be considering only those for which at least one of the coefficients a,b,c,d,e,f,g,h is equal to zero and, simultaneously, not more that 5 of them vanish. The family of all such functions will be divided into the following subclasses:

$$\mathcal{F}_i=\{F(x,y)=\frac{a_1xy+a_2x+a_3y+a_4}{a_5xy+a_6x+a_7y+a_8} \quad : \quad a_i=0, a_n\neq 0, n\neq i\},$$

$$i\in\{1,...,8\};$$

$$\mathcal{F}_{i,j}=\{F(x,y)=\frac{a_1xy+a_2x+a_3y+a_4}{a_5xy+a_6x+a_7y+a_8} \quad : \quad a_i=a_j=0, a_n\neq 0,$$

$$n\in\{1,...,8\}\setminus\{i,j\}\}, \quad i\neq j \quad i,j\in\{1,...,8\};$$

$$\mathcal{F}_{i,j,k} = \{F(x,y) = \frac{a_1xy + a_2x + a_3y + a_4}{a_5xy + a_6x + a_7y + a_8} \ :$$

$$a_i = a_j = a_k = 0, \quad a_n \neq 0, \quad n \in \{1, ..., 8\} \setminus \{i, j, k\}\},$$

$$i, j, k \in \{1, ..., 8\} \quad i, j, k \quad \text{pairwise} \quad \text{different};$$

$$\mathcal{F}_{i,j,k,l} = \{F(x,y) = \frac{a_1xy + a_2x + a_3y + a_4}{a_5xy + a_6x + a_7y + a_8} \ :$$

$$a_i = a_j = a_k = a_l = 0, \quad a_n \neq 0, n \in \{1, ..., 8\} \setminus \{i, j, k, l\}\},$$

$$i, j, k, l \in \{1, ..., 8\}, \quad i, j, k, l \quad \text{pairwise} \quad \text{different};$$

$$\mathcal{F}_{i,j,k,l,m} = \{F(x,y) = \frac{a_1xy + a_2x + a_3y + a_4}{a_5xy + a_6x + a_7y + a_8} \ : \quad a_i = a_j = a_k = a_l =$$

$$a_m = 0, a_n \neq 0, n \in \{1, ..., 8\} \setminus \{i, j, k, l, m\}\}, \quad i, j, k, l, m \in \{1, ..., 8\},$$

$$i, j, k, l, m \quad \text{pairwise} \quad \text{different}.$$

In each of the classes $\mathcal{F}_i$ (clearly we have 8 of them) there are rational functions admitting exacly one zero coefficient from among a, b, c, d, e, f, g, h. In each of the 28 classes $\mathcal{F}_{i,j}$ there are rational functions with exacly 2 zero coeffitients. Analogously, in case where exacly 3 coeffitients occurring in (3) are vanishing, the such a rational function belongs to the family of 55 (not 56 because $\mathcal{F}_{5,6,7} = \mathcal{F}_{5,6,7,8}$) $\mathcal{F}_{i,j,k}$ classes; in case where exacly 4 coeffitients occurring in (3) are vanishing, the such a rational function belongs to the family of 65 (in practice, and 70 theoretically) $\mathcal{F}_{i,j,k,l}$ classes; in case where exacly 5 coeffitients occurring in (3) are vanishing, the such a rational function belongs to the family of 46 (in practice, and 56 theoretically) $\mathcal{F}_{i,j,k,l,m}$ classes. We shall show that only some particular members of the classes spoken of happen to be associative. Moreover, not all elements of a given class are associative. For each of the classes in question we shall formulate a necessary and sufficient condition for the associativity of its members.

Equation (E) for a rational function of the form (3) may equivalently be written as the equality between two three-place rational function or as the equality of two polynomials of second degree in three independent variables. Such a polynomial yields a function of the form

$$W(x, y, z) = \sum_{i,j,k=0}^{2} c_{i,j,k} x^i y^j z^k,$$

where $c_{i,j,k}$ are constants. Thus it is a linear combination of 27 second degree monomials in three variables (the number of 3−element variations with repeatings from a 3−element set). The majority of the function classes considered the equality of the corresponding polynomials forces the coefficients of some monomials involred to be equal to 0. After the comparison of the corresponding coefficients one obtains a system of 27 equalities concerning the parameters a, b, c, d, e, f, g, h, which is equivalent to the validity of equation (E).

Theoretically, that system may be viewed as a necessary and sufficient condition for the associativity of functions of the form (3). In practice, for a concrete function considered with all the coefficients nonvanishing, it is usually much simpler and faster to check its associativity directly with the aid of equation (E).

However, in the case where at least one of the coefficients in question is equal to zero, the system spoken of actually reduces to a smalles number of equations. Even more, for the subsequent classes the question of their solvability may easily be answered. More precisely, one may classify these classes via a selection of those admitting associative operations and to state readable necessary and sufficient conditions equivalent to the associativity property.

A system in question is explicitely written in the statement of the Proposition and proves to be useful in the proofs relevant theorems. This system will be used to simplify the proofs presented. Plainly, each assertion of the theorems estblished may directly be derived from equation (E). Nevertheless, an appeal to this system allows one to avoid numerous and tedious transfomations of (E) which would be necessary otherwise.

The following lemma will prove to be useful in the sequel.

Proposition. *If a rational function given by formula*

$$F(x, y) = \frac{axy + bx + cy + d}{exy + fx + gy + h},$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$, $e^2 + f^2 + g^2 + h^2 > 0$ *is associative, then the following equalities hold true:*

(I)

$$abe^2 + a^2eg + be^2g = a^2ef + ace^2 + ce^2f,$$

(II)

$$\begin{aligned} & a^3f + a^2eh + be^2h + abeg + abef + befg \\ = \ & ade^2 + de^2f + a^2be + a^2f^2 + bce^2 + cef^2, \end{aligned}$$

(III)

$$b^2e^2 + befg + aceg + abeg + beg^2 = acef + abef + cef^2 + c^2e^2 + cefg,$$

(IV)

$$\begin{aligned} & a^2g^2 + bce^2 + beg^2 + a^2ce + ade^2 + de^2g \\ = \ & aceg + acef + cefg + a^3g + a^2eh + ce^2h, \end{aligned}$$

(V)

$$\begin{aligned} & a^2bf + befh + abfg + b^2ef + bf^2g + a^2cf + aceh + 2begh + abeh + adeg \\ = \ & ab^2e + adef + def^2 + bcef + abf^2 + cf^3 + abce + 2cde^2 + defg + cefh, \end{aligned}$$

(VI)

$$\begin{aligned} & 2a^2cf + a^2gh + begh + abg^2 + bcef + bfg^2 + aceh + 2adef \\ = \ & 2adeg + acf^2 + bceg + cf^2g + abeh + 2a^2bg + a^2fh + cefh, \end{aligned}$$

(VII)

$$\begin{aligned} & 2bde^2 + begh + acg^2 + bceg + bg^3 + ac^2e + adeg + deg^2 + abce + defg \\ = \ & aceh + adef + 2cefh + cfg^2 + a^2cg + c^2eg + cegh + a^2bg + acfg + abeh, \end{aligned}$$

(VIII)

$$\begin{aligned} & ab^2f + abfh + bf^2h + beh^2 + a^2df + adeh \\ = \ & b^3e + bdef + df^3 + abde + d^2e^2 + defh, \end{aligned}$$

(IX)

$$abcf + acfh + 2bfgh + b^2eh + adfg = b^2ce + 2cdef + df^2g + abfh + cf^2h,$$

(X)

$$2abcf + abgh + bfgh + bceh + adf^2 = b^2ce + 2bdeg + df^2g + abfh + ab^2g,$$

(XI)

$$ac^2f+acgh+2cdef+bc^2e+dfg^2=2abcg+adg^2+bceh+acfh+cfgh,$$

(XII)

$$acgh+2bdeg+bg^2h+bc^2e+dfg^2=c^2eh+adfg+2cfgh+abcg+abgh,$$

(XIII)

$$\begin{aligned} &acde+e^2d^2+degh+c^3e+cdeg+dg^3\\ &= a^2dg+adeh+ceh^2+ac^2g+acgh+cg^2h, \end{aligned}$$

(XIV)

$$a^2bf+abeh+befh=ab^2e+bde^2+def^2,$$

(XV)

$$acfg+b^2eg+bfg^2=c^2ef+abfg+cf^2g,$$

(XVI)

$$ac^2e+cde^2+deg^2=ac^2g+aceh+cegh,$$

(XVII)

$$\begin{aligned} &abdf+bfh^2+cbfh+bgh^2+2acdf+adgh+b^2cf+bdf^2+cdeh+adfh\\ &= b^2de+bdeh+bcde+2d^2eg+dfgh+b^3g+bdfg+b^2fh+df^2h+abdg, \end{aligned}$$

(XVIII)

$$\begin{aligned} &acdf+ach^2+2bgh^2+2bdeh+adgh+bc^2f+c^2fh+bdfg\\ &= 2cdeh+adfh+2cfh^2+b^2cg+cdfg+b^2gh+abdg+abh^2, \end{aligned}$$

(XIX)

$$\begin{aligned} &acdf+2d^2ef+bcde+dfgh+c^3f+c^2gh+cdfg+dg^2h+c^2de+cdeh\\ &= adgh+2abdg+bdeh+bcgh+adfh+cfh^2+bc^2g+acdg+cdg^2+cgh^2, \end{aligned}$$

(XX)

$$bfh^2 + abdf + adfh = b^2de + d^2ef + df^2h,$$

(XXI)

$$bc^2f + bcgh + cdf^2 = b^2cg + bdg^2 + bcfh,$$

(XXII)

$$c^2de + d^2eg + dg^2h = acdg + adgh + cgh^2,$$

(XXIII)

$$\begin{aligned} & bdfh + bh^3 + ad^2f + adh^2 + bcdf + cdfh \\ = \quad & d^2eh + bd^2e + b^2dg + d^2fg + b^2h^2 + dfh^2, \end{aligned}$$

(XXIV)

$$bcdf + d^2f^2 + c^2df + cdgh + cdfh = bdgh + bdfh + bcdg + d^2g^2 + b^2dg,$$

(XXV)

$$\begin{aligned} & c^2df + c^2h^2 + d^2fg + cd^2e + d^2eh + dgh^2 \\ = \quad & cdgh + bcdg + bdgh + ad^2g + adh^2 + ch^3, \end{aligned}$$

(XXVI)

$$cd^2f + cdh^2 + d^2fh + = d^2gh + bd^2g + bdh^2,$$

(XXVII)

$$\begin{aligned} & a^2df + beh^2 + bfgh + 2ac^2f + acgh + 2bg^2h + abcf + acfh \\ + \quad & adg^2 + b^2ce + 3bdef + df^2g + c^2eh + adfg + acde + degh \\ = \quad & abde + adf^2 + 2cf^2h + bc^2e + 3cdeg + dfg^2 + 2ab^2g \\ + \quad & adfg + b^2eh + acfh + cfgh + abfh + abcg + a^2dg + ceh^2 + defh. \end{aligned}$$

Conversely, any function of form (3) *which satisfies all the conditions* (I)-(XXVII) *fulfills equation* (E).

Proof. Assume that a rational function F given by (3) is associative. We obtain equations (I),(II),...,(XXIV),(XXV) and (XXVII) from equation

$$F(F(x,y),z) = F(x,F(y,z))$$

by comparison of the coefficients of monomials $x^2y^2z^2$, x^2y^2z, x^2yz^2, xy^2z^2, x^2yz, xy^2z, xyz^2, x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, x^2y^2, x^2z^2, y^2z^2, xy, xz, yz, x^2, y^2, z^2, x, y, z, xyz, respectively. The comparison of monomials of degree zero leads to equation (XXVI). Obviously if a function of form (3) fulfills the above conditions, then it is associative. Thus the proof has been completed.

We notice that

Remark. *The system of the equations* (I)-(XXVII) *has at least one solution such that all coefficients are nonvanishing, namely* $a = b = c = d = e = f = g = h = 1$. *However a function defined by*

$$F(x,y) = \frac{xy + x + y + 1}{xy + x + y + 1} = 1$$

fails to be an associative, nontrivial operation of form (3) *with all nonvanishing coefficients.*

It turns out that in the class of rational function of form (3), where exactly one of the eight coefficients is equal zero only the following are associative:

$$F(x,y) = \frac{bex + bey + bf}{e^2xy + efx + efy + be + f^2} \quad \text{with} \quad b, e, f \neq 0, be \neq -f^2;$$

$$F(x,y) = \frac{a^2exy + a^2ex + a^2ey + a^2f}{(a-f)efxy + aefx + aefy} \quad \text{with} \quad a, e, f \neq 0, a \neq f.$$

Namely the following theorem is true:

Theorem 1. *In the class of the rational function of form*

$$F(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h},$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$, $e^2 + f^2 + g^2 + h^2 > 0$ *and exactly one (whichever) of coefficients is equal to zero, only following subclass contains associative functions:*

$$F(x,y) = \frac{bx + cy + d}{exy + fx + gy + h} \quad \text{with} \quad b, c, d, e, f, g, h \neq 0; \tag{1.1}$$

$$F(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy} \quad \text{with} \quad a, b, c, d, e, f, g \neq 0. \tag{1.2}$$

Moreover, operations of form (1.1) *are asociative (on natural domain) iff* $c = b$, $g = f$, $d = \frac{bf}{e}$, $h = \frac{be+f^2}{e}$ *and operations of form* (1.2) *are asociative (on natural domain) iff* $c = b$, $g = f$, $d = \frac{af}{e}$, $e = \frac{af-f^2}{a}$.

Proof. First we show that no one of the functions is associative:

$$F(x, y) = \frac{axy + cy + d}{exy + fx + gy + h} \quad \text{with} \quad a, c, d, e, f, g, h \neq 0; \tag{1.3}$$

$$F(x, y) = \frac{axy + bx + d}{exy + fx + gy + h} \quad \text{with} \quad a, b, d, e, f, g, h \neq 0; \tag{1.4}$$

$$F(x, y) = \frac{axy + bx + cy}{exy + fx + gy + h} \quad \text{with} \quad a, b, c, e, f, g, h \neq 0; \tag{1.5}$$

$$F(x, y) = \frac{axy + bx + cy + d}{fx + gy + h} \quad \text{with} \quad a, b, c, d, f, g, h \neq 0; \tag{1.6}$$

$$F(x, y) = \frac{axy + bx + cy + d}{exy + gy + h} \quad \text{with} \quad a, b, c, d, e, g, h \neq 0; \tag{1.7}$$

$$F(x, y) = \frac{axy + bx + cy + d}{exy + fx + h} \quad \text{with} \quad a, b, c, d, e, f, h \neq 0. \tag{1.8}$$

Assume the contrary: suppose that operation (1.3) is associative. By Proposition equality (XIV) hold true. In case $b = 0$ this equation gives $def^2 = 0$, which is impossible. For operation (1.4) (case $c = 0$) equation (XVI) gives $deg^2 = 0$; for (1.5) (case $d = 0$) equation (XX) gives $bfh^2 = 0$; for (1.6) ($e = 0$) by (XIV) we have $a^2bf = 0$; for (1.7) ($f = 0$) by (XV) we have $b^2eg = 0$ and for (1.8) (case $g = 0$) by (XXII) we obtain $c^2de = 0$. Therefore, all these cases lead to a contradiction.

Let function of form (1.1) be associative. By Proposition equalities (I),..., (XXVII) are satisfied. On setting $a = 0$ in equation (I) we see that

$$cf = bg \tag{i}$$

and obviously

$$cf^2h = bfgh. \tag{ii}$$

By (IX) (with $a = 0$)we obtain

$$2bfgh + b^2eh = b^2ce + 2cdef + df^2g + cf^2h,$$

which, by (ii), gives

$$bfgh = b^2ce + 2cdef + df^2g - b^2eh.$$

However, by (X) (for $a = 0$) we have

$$bfgh = b^2ce + 2bdeg + df^2g - bceh.$$

Therefore, by (i) (which gives $2cdef = 2bdeg$)

$$bceh = b^2eh,$$

i.e. $c = b$ and, by (i), $g = f$. Thus we have

$$c = b, \qquad g = f. \tag{iii}$$

Further, by (IV) (with $a = 0$) and (iii),

$$b^2 + df = bh, \tag{iv}$$

i. e.

$$b^2ef + def^2 = befh. \tag{v}$$

By (XVI) and (iii) we have

$$bde^2 + def^2 = befh.$$

Therefore, on account of (v) one has

$$bde^2 = b^2ef,$$

i.e.

$$de = bf. \tag{vi}$$

Further, by (iv) we obtain

$$h = \frac{b^2 + df}{b}$$

and by (vi)

$$d = \frac{bf}{e},$$

whence

$$h = \frac{b^2 + \frac{bf^2}{e}}{b} = b + \frac{f^2}{e} = \frac{be + f^2}{e}.$$

This means that if function from class (1.1) is associative it must be of the form

$$F(x,y) = \frac{bex + bey + bf}{e^2xy + efx + efy + be + f^2} \qquad \text{with} \qquad b, e, f \neq 0, be \neq f^2.$$

A direct calculation shows that this function fullfils equation (E).

Now we assume, that function of form (1.2) is associative. By Proposition equalities (I),..., (XXVII) hold true. On setting $h = 0$ in equation (XXVI) we see that

$$cf = bg \tag{$\star$}$$

i.e.

$$bc^2f = b^2cg \tag{$\star\star$}$$

and from (XXI) (with $h = 0$), by ($\star\star$), we have

$$cdf^2 = bdg^2,$$

whence by ($\star$) we get $g = f$ and from ($\star$) $c = b$. So we have

$$c = b, \qquad g = f. \tag{$\star\star\star$}$$

Further, by (XXIII) (for $h = 0$) and ($\star\star\star$), we see that

$$af = be + f^2. \tag{$\bullet$}$$

Putting $h = 0$ in (XVI) and using ($\star\star\star$) we have

$$ab^2e + bde^2 + def^2 = ab^2f.$$

Therefore, by ($\bullet$)

$$ab^2e + (af - f^2)de + def^2 = ab^2f$$

i.e.

$$b^2e + def = b^2f$$

and whence

$$b^2de + d^2ef = b^2df. \tag{$\bullet\bullet$}$$

Further, by (XXII) (with $h = 0$) and ($\star\star\star$) we obtain

$$b^2de + d^2ef = abdf.$$

Therefore by (••)

$$abdf = b^2 df$$

i.e.

$$b = a. \qquad (\bullet\bullet\bullet)$$

Putting $h = 0$ in (XIV) and applying $(\bullet\bullet\bullet)$ we have

$$a^3 f = a^3 e + ade^2 + def^2$$

which jointly with $(\bullet)$ and $(\bullet\bullet\bullet)$ leads to

$$a^2(ae + f^2) = a^3 e + de(af - f^2) + def^2.$$

Therefore

$$a^2 f^2 = adef$$

which implies that

$$d = \frac{af}{e}.$$

By $(\bullet)$ and $(\bullet\bullet\bullet)$ we get

$$e = \frac{af - f^2}{a}.$$

This means, that if function from class (1.2) is associative then it must be of the form

$$F(x,y) = \frac{axy + ax + ay + \frac{af}{e}}{\frac{af-f^2}{a}xy + fx + fy} \qquad \text{with} \qquad a, e, f \neq 0, a \neq f$$

i.e.

$$F(x,y) = \frac{a^2exy + a^2ex + a^2ey + a^2f}{(a-f)efxy + aefx + aefy} \qquad \text{with} \qquad a, e, f \neq 0, a \neq f$$

which was to be shown. It is easily to check, that the latter operation is associative (on natural domain) whenever $a, e, f \neq 0, a \neq f$. Thus the proof has been completed.

Now we distinguish such subclasses of the class of functions of form (3) for which exactly two coefficients are equal zero. The corresponding result reads as follows:

Theorem 2. *In the class of rational functions of the form*

$$F(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h},$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$, $e^2 + f^2 + g^2 + h^2 > 0$ *and exactly two (whichever) coefficients are equal zero, no one is associative.*

Proof. We shall consider 28 cases (the number of choises of exactly two elements from among eight). In each case we will assume (for the indirect proof) that a rational function with exactly two wanishing coefficients is associative and we will use Proposition to get a contradiction. In case $a = b = 0$ by (I) we obtain

$$ce^2 f = 0,$$

which is impossible. Similary,
if $b = e = 0$, (XVI) gives $ac^2 g = 0$;
if $e = h = 0$, (XX) gives $abdf = 0$;
if $f = h = 0$, (XXVI) gives $bd^2 g = 0$;
if $a = c = 0$, (I) gives $b^2 eg = 0$;
if $a = d = 0$, (XVI) gives $cegh = 0$;
if $a = f = 0$, (I) gives $be^2 g = 0$;
if $a = g = 0$, (I) gives $ce^2 f = 0$;
if $b = d = 0$, (XXII) gives $cgh^2 = 0$;
if $b = c = 0$, (XVI) gives $deg^2 = 0$;
if $b = h = 0$, (XIV) gives $def^2 = 0$;
if $b = g = 0$, (XIV) gives $def^2 = 0$;
if $c = d = 0$, (XX) gives $bfh^2 = 0$;
if $c = e = 0$, (XIV) gives $a^2 bf = 0$;
if $c = h = 0$, (XVI) gives $deg^2 = 0$;
if $c = f = 0$, (XVI) gives $deg^2 = 0$;
if $d = e = 0$, (XX) gives $bfh^2 = 0$;
if $d = f = 0$, (XV) gives $b^2 eg = 0$;
if $d = g = 0$, (XX) gives $bfh^2 = 0$;
if $e = f = 0$, (XVI) gives $ac^2 g = 0$;
if $f = g = 0$, (XX) gives $b^2 de = 0$;
if $g = h = 0$, (XXVI) gives $cd^2 f = 0$;
if $e = g = 0$, (XXI) gives $bc^2 f + cdf^2 = bcfh$,
whence

$$bc + df = bh$$

whereas (X) gives $2abcf + adf^2 = abfh$, whence $2bc + df = bh$. Therefore

$$bc = 0.$$

In the remaining cases we proceed as follows.

Let $d = h = 0$. By (XVI) we get $ac^2e = ac^2g$ whence

$$g = e. \tag{i}$$

Putting $d = h = 0$ in (XIX) we obtain $c^3f = bc^2g$, i.e.

$$cf = bg. \tag{ii}$$

By (XIV) (with $d = h = 0$) we have $a^2bf = ab^2e$, i.e.

$$af = be. \tag{iii}$$

By $(i), (ii), (iii)$ we get $af = cf$, whence

$$c = a. \tag{iv}$$

On setting $d = h = 0$ in (VII) and using $(i), (iv)$ we see that

$$abe^2 + be^3 = afe^2 + a^2e^2$$

i.e.

$$ab + be = af + a^2.$$

By (iii) this means that $b = a$ and, by (iii), $f = e$. So we have

$$b = a, \quad c = a, \quad f = e, \quad g = e. \tag{v}$$

Therefore, in this case, a rational function considered has to have form

$$F(x, y) = \frac{axy + ax + ay}{exy + ex + ey} = \frac{a}{e},$$

which is impossible, because merely exactly two coefficients are equal zero.

Let $c = g = 0$. By (XIII) we get $d^2e^2 = adeh$, whence

$$de = ah$$

and by (IX) we have $b^2eh = abfh$, i.e.

$$be = af.$$

Therefore, in this case, we infer that

$$F(x,y) = \frac{axy+bx+d}{exy+fx+h} = \frac{aexy+bex+de}{e^2xy+efx+eh} =$$
$$= \frac{aexy+afx+ah}{e^2xy+efx+eh} = \frac{a}{e}\frac{exy+fx+h}{exy+fx+h} = \frac{a}{e}.$$

This is contradiction because merely two coefficients are equal zero.

Let now $b = f = 0$. From (I) we derive the equality

$$ag = ce$$

and therefore, by (V), we have $aceh + cede = 2cde^2$, i.e.

$$ah = de.$$

Thus, in this case, we obtain

$$F(x,y) = \frac{axy+cy+d}{exy+gy+h} = \frac{a^2xy+acy+ad}{aexy+agy+ah} =$$
$$= \frac{a^2xy+acy+ad}{aexy+cey+de} = \frac{a(axy+cy+d)}{e(axy+cy+d)} = \frac{a}{e},$$

which again is contradiction because merely two coefficients are equal zero.

Let now $a = e = 0$. From (V) and (VIII) we see that

$$bg = cf$$

and

$$bh = df.$$

Therefore, in this case, we get

$$F(x,y) = \frac{bx+cy+d}{fx+gy+h} = \frac{b^2x+bcy+bd}{bfx+bgy+bh} =$$
$$= \frac{b^2x+bcy+bd}{bfx+cfy+df} = \frac{b(bx+cy+d)}{f(bx+cy+d)} = \frac{b}{f},$$

a similar contradiction.

Let finally $a = h = 0$. We observe that in this case (I) gives $be^2g = ce^2f$ whence

$$bg = cf. \tag{$\star$}$$

By (II) (with $a = h = 0$) we get $befg = de^2 + bce^2 + cef^2$, which by $(\star)$ leads to $de^2f + bce^2 = 0$, i.e.

$$df = -be. \tag{$\star\star$}$$

Putting $a = h = 0$ in (IV) we obtain $bce^2+beg^2+de^2g = cefg$, whence by $(\star)$ $bce^2 + de^2g = 0$, i.e.

$$bc = -dg. \tag{$\star\star\star$}$$

Now by $(\star\star)$ and $(\star\star\star)$ we have $dg = df$ i.e.

$$g = f$$

and by $(\star)$

$$c = b.$$

Further, by (IV) (for $a = h = 0$, $b = c$, $g = f$,) we conclude $b^2+de^2f = 0$, i.e. $b^2 = -df$. Therefore, by $(\star\star)$ we have $b^2 = be$ getting

$$e = b.$$

After setting $a = h = 0$, $e = c = b$, $g = f$ in (V), we have $bdf^2 = -b^3d$, whence

$$f^2 = -b^2.$$

This contradiction finishes the proof of the theorem.

In the sequel, we consider such subclasses of class of functions of the form (3), where exactly three coefficients are equal zero. Simultaneously, the case $e = f = g = 0$ is numbered among cases, where four coefficients vanish ($e = f = g = h = 0$), because

$$\frac{axy + bx + cy + d}{h} = Axy + Bx + Cy + D,$$

where $A = \frac{a}{h}$, $B = \frac{b}{h}$, $C = \frac{c}{h}$, $D = \frac{d}{h}$.

We are going to show, that only the following functions from these classes are associative:

$$F(x, y) = \frac{aexy}{e^2xy + efx + efy + f^2 - af} \qquad \text{with} \qquad a, e, f \neq 0, f \neq a;$$

$$F(x, y) = \frac{axy + bx + by}{exy + b} \qquad \text{with} \qquad a, b, e \neq 0;$$

$$F(x,y) = \frac{axy+d}{ax+ay+h} \qquad \text{with} \qquad a,d,h \neq 0.$$

More exactly we have

Theorem 3. *In the class of the rational functions of the form*

$$F(x,y) = \frac{axy+bx+cy+d}{exy+fx+gy+h},$$

where $a,b,c,d,e,f,g,h \in \mathbb{R}$, $e^2+f^2+g^2+h^2 > 0$ *and exactly three (whichever) of these coefficients are equal to zero, merely following subclass cointains associative functions:*

$$F(x,y) = \frac{axy}{exy+fx+gy+h}, \qquad \text{with} \qquad a,e,f,g,h \neq 0; \qquad (3.1)$$

$$F(x,y) = \frac{axy+bx+cy}{exy+h}, \qquad \text{with} \qquad a,b,c,e,h \neq 0; \qquad (3.2)$$

$$F(x,y) = \frac{axy+d}{fx+gy+h}, \qquad \text{with} \qquad a,d,f,g,h \neq 0. \qquad (3.3)$$

Moreover, operations of form (3.1) *are asociative (on natural domain) iff* $g=f$, $h=\frac{f^2-af}{e}$; *operations of form* (3.2) *are asociative (on natural domain) iff* $h=c=b$ *and operations of form* (3.3) *are asociative (on natural domain) iff* $g=f=a$.

Proof. First we consider classes (3.1),(3.2),(3.3). Assume that function of form (3.1) is associative. By Proposition we have equalities (I),...,(XXVII) (with $b=c=d=0$). From (I) we conclude that $a^2eg = a^2ef$, i.e.

$$g = f$$

and, by II, $a^3f + a^2eh = a^2f^2$, whence

$$h = \frac{f^2-af}{e}.$$

Therefore

$$F(x,y) = \frac{axy}{exy+fx+fy+\frac{f^2-af}{e}},$$

which is associative (a straightforward verification), which was to be shown. Now let a function of form (3.2) be associative. By Proposition we conclude that equalities (I),...,(XXVII) (with $d = f = g = 0$) hold true. From (I) we see that $abe^2 = ace^2$, i.e.

$$c = b$$

and, by (XIV), $abeh = ab^2e$, whence

$$h = b.$$

Therefore

$$F(x, y) = \frac{axy + bx + by}{exy + b},$$

which is associative (a straightforward verification), which was to be shown. Further let function of form (3.3) be associative. By Proposition we have equalities (I),...,(XXVII) (with $b = c = e = 0$). From (II) we obtain $a^3f = a^2f^2$, whence

$$f = a$$

and, by (IV), $a^2g^2 = a^3g$, showing that

$$g = a.$$

Therefore

$$F(x, y) = \frac{axy + d}{ax + ay + h}.$$

A simple calculation shows that F is associative.

To show that no one from the remaining classes is associative functions it suffices to observe that
for $b = c = f = 0$, by (I), we get $a^2eg = 0$;
for $b = c = g = 0$, by (I), we get $a^2ef = 0$;
for $b = c = h = 0$, by (XIV), we get $def^2 = 0$;
for $b = d = e = 0$, by (XXII), we get $cgh^2 = 0$;
for $b = d = f = 0$, by (XXII), we get $cgh^2 = 0$;
for $b = d = g = 0$, by (XII), we get $c^2eh = 0$;
for $b = d = h = 0$, by (XXVII), we get $ac^2f = 0$;
for $b = f = g = 0$, by (XXII), we get $c^2de = 0$;
for $b = f = h = 0$, by (II), we get $ade^2 = 0$;

for $b = g = h = 0$, by (IX), we get $cdef = 0$;
for $c = d = e = 0$, by (XX), we get $bfh^2 = 0$;
for $c = d = f = 0$, by (XV), we get $b^2eg = 0$;
for $c = d = g = 0$, by (X), we get $abfh = 0$;
for $c = d = h = 0$, by (XXVII), we get $ab^2g = 0$;
for $c = f = g = 0$, by (I), we get $abe^2 = 0$;
for $c = f = h = 0$, by (XI), we get $adg^2 = 0$;
for $c = g = h = 0$, by (XIII), we get $d^2e^2 = 0$;
for $d = f = h = 0$, by (XIV), we get $ab^2e = 0$;
for $d = g = h = 0$, by (XV), we get $c^2ef = 0$;
for $f = g = h = 0$, by (XX), we get $b^2de = 0$;
for $a = c = d = 0$, by (XIV), we get $befh = 0$;
for $a = b = d = 0$, by (XVI), we get $cegh = 0$;
for $a = b = c = 0$, by (XIV), we get $def^2 = 0$;
for $a = b = g = 0$, by (XIV), we get $def^2 = 0$;
for $a = b = f = 0$, by (XVIII), we get $cdeh = 0$;
for $a = b = h = 0$, by (XIV), we get $def^2 = 0$;
for $a = c = e = 0$, by (XV), we get $bfg^2 = 0$;
for $a = c = f = 0$, by (I), we get $be^2g = 0$;
for $a = c = g = 0$, by (III), we get $b^2e^2 = 0$;
for $a = c = h = 0$, by (I), we get $be^2g = 0$;
for $a = d = e = 0$, by (V), we get $cf^3 = 0$;
for $a = d = f = 0$, by (I), we get $be^2g = 0$;
for $a = d = g = 0$, by (XX), we get $bfh^2 = 0$;
for $a = d = h = 0$, by (IX), we get $b^2ce = 0$;
for $a = e = f = 0$, by (VII), we get $bg^3 = 0$;
for $a = e = g = 0$, by (V), we get $cf^3 = 0$;
for $a = e = h = 0$, by (V), we get $cf^3 = 0$;
for $a = g = h = 0$, by (I), we get $ce^2f = 0$;
for $a = f = g = 0$, by (XX), we get $b^2de = 0$;
for $a = f = h = 0$, by (I), we get $be^2g = 0$;
for $c = e = f = 0$, by (XI), we get $adg^2 = 0$;
for $b = e = f = 0$, by (VI), we get $a^2gh = 0$;
for $c = e = g = 0$, by (XXV), we get $adh^2 = 0$;
for $b = e = g = 0$, by (X), we get $adf^2 = 0$;
for $c = e = h = 0$, by (XXI), we get $bdg^2 = 0$;
for $b = e = h = 0$, by (XVI), we get $ac^2g = 0$;
for $d = e = f = 0$, by (XVI), we get $ac^2g = 0$;
for $d = e = g = 0$, by (XX), we get $bfh^2 = 0$;

for $d = e = h = 0$, by (XVI), we get $ac^2g = 0$;
for $e = f = h = 0$, by (XVI), we get $ac^2g = 0$;
for $e = g = h = 0$, by (XIV), we get $a^2bf = 0$;
for $a = b = e = 0$, by (XX), we get $df^2h = 0$.

So, in each of the remaining cases we obtain contradiction with the assumption that exactly three coefficients are equal to zero. This ends the proof.

Using similar metods we can determine associative functions of form (3) with exactly four wanishing coefficients. Clearly the case $e = f = g = h = 0$ is excluded and because

$$F(x,y) = \frac{bx + cy + d}{h} = Bx + Cy + D, \qquad \text{with} \qquad b, c, d, h \neq 0;$$

$$F(x,y) = \frac{axy + bx + cy}{h} = Axy + Bx + Cy, \qquad \text{with} \qquad a, b, c, h \neq 0;$$

$$F(x,y) = \frac{axy + bx + d}{h} = Axy + Bx + D, \qquad \text{with} \qquad a, b, d, h \neq 0;$$

$$F(x,y) = \frac{axy + cy + d}{h} = Axy + Cy + D, \qquad \text{with} \qquad a, b, c, h \neq 0,$$

where $A = \frac{a}{h}$, $B = \frac{b}{h}$, $C = \frac{c}{h}$, $D = \frac{d}{h}$; the cases where $a = e = f = g = 0$, $d = e = f = g = 0$, $c = e = f = g = 0$, $b = e = f = g = 0$ are concidered in classes with 5 vanishing coefficients.

It is not hard to verify the following

Theorem 4. *In the class of the rational functions of the form*

$$F(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h},$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$, $e^2 + f^2 + g^2 + h^2 > 0$ *and exactly four (whichever) of the coefficients are equal zero, merely the following subclass contains associative functions:*

$$F(x,y) = \frac{axy}{exy + fx + gy}, \qquad \text{with} \qquad a, e, f, g \neq 0; \tag{4.1}$$

$$F(x,y) = \frac{axy}{fx + gy + h}, \qquad \text{with} \qquad a, f, g, h \neq 0; \tag{4.2}$$

$$F(x,y)=\frac{bx+cy}{exy+h}, \qquad \text{with} \qquad b,c,e,h\neq 0. \tag{4.3}$$

$$F(x,y)=\frac{axy+d}{fx+gy}, \qquad \text{with} \qquad a,d,f,g\neq 0. \tag{4.4}$$

$$F(x,y)=Axy+Bx+Cy+D, \qquad \text{with} \qquad A,B,C,D\neq 0. \tag{4.5}$$

Moreover, operations of form (4.1) *are asociative (on natural domain) iff* $g=f=a$; *operations of form* (4.2) *are asociative (on natural domain) iff* $g=f=a$; *operations of form* (4.3) *are asociative (on natural domain) iff* $h=c=b$; *operations of form* (4.4) *are asociative (on natural domain) iff* $g=f=a$ *and operations of form* (4.5) *are asociative (on natural domain) iff* $C=B$, $D=\frac{B(B-1)}{A}$, $B\neq 1$.

Analogously, we can study the class of rational functions of form (3) with exactly three nonvanishing coefficients and to prove the following

Theorem 5. *In the class of the rational functions of the form*

$$F(x,y)=\frac{axy+bx+cy+d}{exy+fx+gy+h},$$

where $a,b,c,d,e,f,g,h\in\mathbb{R}$, $e^2+f^2+g^2+h^2>0$ *and exactly five (whichever) of the coefficients are equal to zero, merely the following subclass contains associative functions:*

$$F(x,y)=\frac{axy}{fx+gy}, \qquad \text{for} \qquad a,f,g\neq 0; \tag{5.1}$$

$$F(x,y)=axy+bx+cy, \qquad \text{for} \qquad a,b,c\neq 0. \tag{5.2}$$

Moreover, operations of form (5.1) *are asociative (on natural domain) iff* $g=f=a$; *operations of form* (5.2) *are asociative iff* $c=b=1$.

The only remaining case where at most 2 of the coefficients in question do not vanish is trivial.

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