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Some Applications of Cellular Automata in Learning Systems Constructions

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Abstract
Today the von Neumann invention of cellular automata is one of important tools of artificial intelligence researches. Cellular automata show their usage in simultaneous calculations, creating data bases and simulating physical processes. The goal of this article is to introduce to the learning system called CELLS, which manner of work is directly connected with the cellular automata evolution rules.

1 Introduction

The CELLS system is a type of a learning system with supervision. Knowledge in alike systems is isolated from the set of vector pairs $\langle W_i, W_o \rangle$, in which the first elements (W_i) indicate questions for a student, the second ones (W_o) are desired answers for the asked questions. Starting from the analysis of the above finite set of vector pairs the system has to create an algorithm of giving correct answers for as big as possible set of probable questions, also those, which have not occurred in the training set.

Let the training pair be denoted as structure of two m-elements sequences

$$P_T = \langle W_i, W_o \rangle = \langle \langle c_{i_1}, c_{i_2}, \dots, c_{i_m} \rangle, \langle c_{o_1}, c_{o_2}, \dots, c_{o_m} \rangle \rangle,$$

where $c_{i_j}, c_{o_j} \in C$ and C is a finite nonempty set. The main idea of the learning systems CELLS is that the sequences W_i and W_o are treated within it as neighbouring lines of some one-dimensional cellular automaton. It is assumed that:

- only sequences from one given test, which means from one given pair $\langle W_i, W_o, \rangle$ are in the nearest distance with each other;
- the sequence W_i precedes the sequence W_o (and not the reverse);
- the distance between different tests (different pairs $\langle W_i, W_o \rangle$) from the training set is arbitrary (unknown).

The aim of the CELLS system is finding such a cellular automaton which evolution rules will change the sequence W_i into the sequence W_o being adequate to it. A learning system based on cellular automata construction distinguishes itself by:

- ability of creating simple algorithms which change input sequences into output ones;
- resistance from noise;
- possibility of simultaneous converting the input sequences of different length limited only by computer counting power.

2 The basic theorems and definitions

2.1 One-dimensional cellular automaton

One-dimensional cellular automaton CA is most often defined as a pair

$$CA = \langle C, f_{CA} \rangle, \tag{1}$$

where C is any nonempty finite set, f_{CA} is a function which determines evolution rules CA:

$$f_{CA}(L_k) = L_{k+1}, \quad L_k = \langle c_{k_1}, c_{k_2}, \dots, c_{k_m} \rangle, \quad c_{k_i} \in C. \tag{2}$$

After giving the initial configuration L_0 the action of one-dimensional cellular automaton consists in cyclically replacing the system L_k by the next one L_{k+1} according to evolution rules determined by the function f_{CA} [5].

Definition 2.1 Let the initial configuration L_0 and evolution rules f_{CA} be given. The first n lines of evolution of one-dimensional cellular automaton from the system L_0 will be called the *CA product*.

The function f_{CA} is called a *global rule*, unlike the function g_{CA} called a *local rule*. The function g_{CA} determines transformation of the cell c_{k_i} from the system L_k into the cell $c_{(k+1)_i}$ from the system L_{k+1} according to c_{k_i} and a state of neighbouring cells. It is specific for the cellular automata to apply one rule g_{CA} to every cell of the configuration L_k. The local rule g_{CA} combined with the concept of its cyclical use for each cell L_k form global rule f_{CA}. The function f_{CA} can be denoted as

$$\begin{aligned} c_{(k+1)_1} &= g_{CA}(c_{k_1}, c_{k_2}, \ldots, c_{k_{m-1}}, c_{k_m}), \\ c_{(k+1)_2} &= g_{CA}(c_{k_2}, c_{k_3}, \ldots, c_{k_m}, c_{k_1}), \\ &\vdots \\ c_{(k+1)_m} &= g_{CA}(c_{k_m}, c_{k_1}, \ldots, c_{k_{m-2}}, c_{k_{m-1}}), \end{aligned} \tag{3}$$

where $c_{k_i} \in L_k$.

Fact 2.1 Let a cellular automaton be given

$$CA = \langle C, f_{CA} \rangle,$$

let $D(f_{CA})$ be the domain of f_{CA} and let a set $A \subset D(f_{CA})$ be given. The set F created on the CA product

$$F = \bigcup_{0 \le k < n-2} \langle L_k, L_{k+1} \rangle$$

is the function f_{CA} restricted in the domain to a set

$$A = \{L_0, L_1, \ldots, L_{n-2}\}.$$

2.2 Theorem about CA product

Let the nonempty set C, natural number m and m-tuple $L \in C^m$ be given, where C^m is the Cartesian product of m copies of the set C, i.e.

$$L = \langle c_1, c_2, \ldots, c_m \rangle, \quad \text{where} \quad c_i \in C, \quad i \in \{1, 2, \ldots, m\}.$$

The n-tuple sequence $(n > 1)$ will be denoted by C_A

$$C_A = \langle L_0, L_1, \dots , L_{n-1}\rangle, \tag{4}$$

where

$$L_k = \langle c_{k_1}, c_{k_2}, \dots , c_{k_m}\rangle, \quad k \in \{0, 1, \dots , n-1\}.$$

It is obvious that

$$C_A \in \underbrace{C^m \times C^m \times \ldots \times C^m}_{\text{n times}}, \tag{5}$$

Let L_{k_p} mean a cyclic displacement of L_k with p steps to left

$$L_{k_p} = \langle c_{k_{p+1}}, c_{k_{p+2}}, \dots , c_{k_m}, c_{k_1}, \dots , c_{k_p}\rangle, \tag{6}$$

where $0 \leq p < m$. Let the following pair be given:

$$F_{k_p} = \langle L_{k_p}, c_{(k+1)_p}\rangle, \;\; (0 \leq k < n-1). \tag{7}$$

The set of all such pairs will be marked by F^*:

$$F^* = \bigcup_{\substack{0 \leq k < n-1 \\ 0 \leq p < m}} F_{k_p}. \tag{8}$$

It should be noted that F^* is a relation in the Cartesian product of C^m and C:

$$F^* \subset C^m \times C.$$

Theorem 2.1 The sequence C_A is a product of one-dimensional cellular automaton if set F^* is a function*, i.e. the following property holds

$$\bigwedge_{\substack{0 \leq k < n-1 \\ 0 \leq k^* < n-1}} \; \bigwedge_{\substack{0 \leq p < m \\ 0 \leq p^* < m}} L_{k_p} = L_{{k^*}_{p^*}} \rightarrow c_{(k+1)_p} = c_{(k^*+1)_{p^*}}. \tag{9}$$

Proof: The proof results from the method of constructing the set F^* and from fact 2.1.

*In all cases concerning the set F^* the notion of a function will be used from the viewpoint of the set theory.

2.3 Minimization of the set F^*

Definition 2.2 Let set $Z = \{0, 1\}$ be given and let

$$N_Z = \langle z_0, z_1, \ldots, z_m \rangle, \quad z_i \in Z \tag{10}$$

mean m-tuple ordered structure called further a *neighbourhood*.

A function mapping m-tuple L_k into m'-tuple $L_{k|N_Z}$ $(m' \leq m)$ will be called the *reduction* L_k with respect to the neighbourhood N_Z. Only those elements from L_k for which elements from N_Z with the same index have the value 1 will belong to $L_{k|N_Z}$.

The set F^* consisting of pairs $F_k = \langle L_{k|N_Z}, c_{(k+1)} \rangle$ will be denoted as $F^{*|N_Z}$. The number of ones in the neighbourhood N_Z will be called its power. It appears from the definition mentioned above that $F^{*|N_Z} = F^*$ if and only if the neighbourhood N_Z is composed only from ones.

Definition 2.3 A process of finding a set $F^{*|N_Z}$ of the smallest power still being a function will be called *minimization* of the set F^*.

The neighbourhood N_Z is a notion that occurs in the theorem of cellular automata in connexion with local rule notion. The value of the neighbourhood $z_i = 1$ means that the state of the i-th neighbour of the current cell has an influence on the state of the resultant cell*.

Fact 2.2 The minimization F^* refers to a search of the local rule g_{CA} for the cellular automaton which product was used to create the set F^*.

3 Theorem about neighbourhood

Definition 3.1 Let the Boolean function $f : \{0, 1\}^m \to \{0, 1\}^n$ of the following form

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_m) \\ y_2 &= f_2(x_1, x_2, \ldots, x_m) \\ &\vdots \\ y_n &= f_n(x_1, x_2, \ldots, x_m) \end{aligned} \tag{11}$$

be given.

*See formula (3).

A matrix of a *dependence* function f (denoted as $B(f)$) will be called a matrix of $m \times n$ dimension $[b_{ij}]$ with elements

$$b_{ij} = \begin{cases} 0, & \text{if } f_i \text{ is independent from } x_j, \\ 1, & \text{if } f_i \text{ is dependent from } x_j. \end{cases} \tag{12}$$

Fact 3.1 The neighbourhood N_Z minimizing the set F^* of one-dimensional cellular automaton product is identical to the first row of the matrix of dependence function f_{CA}.

Definition 3.2 A matrix of $m \times n$ dimension $[d_{ij}]$ with elements

$$d_{ij} = \frac{\partial f_i}{\partial x_j} = f_i(x_{0_1}, \ldots, x_{0_j}, \ldots, x_{0_m}) \oplus f_i(x_{0_1}, \ldots, \bar{x_{0_j}}, \ldots, x_{0_m}), \tag{13}$$

will be called the Boolean partial derivative [1] of a function f in a form (11) at a point $\mathbf{x} = \langle x_{0_1}, x_{0_2}, \ldots, x_{0_m} \rangle$. Here $\oplus$ stands for exclusive or (XOR) operation, $\bar{x}$ means negation of x.

Let a function f have a form $f : \{0,1\}^m \rightarrow \{0,1\}$ with a domain $D(f)$ and let vectors $\mathbf{x}, \mathbf{y} \in D(f)$ be given. According to definition the i-th element of the derivative of the function f is equal 1 if for two vectors x and y having different i-th components the values $f(\mathbf{x})$ and $f(\mathbf{y})$ are different.

Definition 3.1 Let the cellular automaton product C_A of n lines and m elements in line be given and let a set of pairs*

$$F^* = \bigcup_{\substack{0 \leq i < n-1 \\ 0 \leq p < m}} F_{i_p}, \tag{14}$$

created on it be given, where

$$F_{i_p} = \langle L_{i_p}, c_{(i+1)_p} \rangle, \quad L_{i_p} = \langle c_{i_{p+1}}, c_{i_{p+2}}, \ldots, c_{i_m}, c_{i_1}, \ldots, c_{i_p} \rangle, \tag{15}$$

c_i are elements from th set $\{0,1\}$. Let a natural number q from the closed interval $[1, m]$ be given.

If in the set F^* exist two pairs such that

$$\langle \langle x_1, x_2, \ldots, x_q, \ldots, x_m \rangle,\ a \rangle, \quad \langle \langle y_1, y_2, \ldots, y_q, \ldots, y_m \rangle,\ b \rangle,$$

*See (6), (7) and (8).

for which

$$\bigwedge_{\substack{1<i<m\\ i\neq q}} x_i = y_i, \quad x_q \neq y_q \ \text{ i } \ a \neq b, \tag{16}$$

then the neighbourhood element N_Z with index q is equal 1.

Proof: Let the Boolean derivative of a function $f : \{0,1\}^m \rightarrow \{0,1\}^n$ at a point $\mathbf{x} \in \{0,1\}^m$ be denoted as $f'(\mathbf{x})$. Let $\mathbf{x}$ run through the domain $D(f)$ of a function f. Under the above assumptions it is proved that [4]

$$Sup\{f'(x)\} = B(f), \tag{17}$$

where $B(f)$ means a matrix of the dependence function f and Sup means supremum of Boolean derivative matrices set*. The relation F^* created on the CA product fulfils the condition (9) and it should be identified with a function of the form $f : \{0,1\}^m \rightarrow \{0,1\}$. Therefore, there exist two such pairs $F_j = \langle L_j, c_j \rangle$ and $F_k = \langle L_k, c_k \rangle$ for which the systems L_j and L_k differ only in the q-th component of the Boolean derivative at the point L_j

$$F'^*(L_j) = c_j \oplus c_k. \tag{18}$$

In the case of function $f : \{0,1\}^m \rightarrow \{0,1\}$ the matrix of the dependences function f as well as the Boolean partial derivative have the dimension $m \times 1$. Denoting the elements of $B(F^*)$ as b_i, the elements of $F'^*(L_j)$ as a_i, according to property (17) we have

$$a_i = 1 \rightarrow b_i = 1, \quad 1 \leq i \leq m. \tag{19}$$

Fact (3.1) establishes a relation between the matrix $B(f) = [b_{ij}]$ of the dependences function f and the neighbourhood N_Z in the following way:

$$b_{1q} = 1 \rightarrow N_Z = \langle z_1, z_2, \ldots, z_{q-1}, 1, z_{q+1}, \ldots, z_m \rangle. \tag{20}$$

Because $B(F^*)$ is a vector, it holds

$$b_q = 1 \rightarrow N_Z = \langle z_1, z_2, \ldots, z_{q-1}, 1, z_{q+1}, \ldots, z_m \rangle \tag{21}$$

*Let a set of n matrixes $M = \{M_1, \ldots, M_n\}$ of the same dimension with elements $m_{ij} \in \{0,1\}$ be given. Supremum of M is defined as follows:

$$Sup\{M\} = M_1 \vee M_2 \vee \ldots \vee M_n,$$

where the operator $\vee$ means the Boolean sum.

that ends the proof.

Example 3.1 Let $C = \{0, 1\}$, $m = 5$ and let a set F^* be created on the CA product be given:

$$C_A : \begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array},$$

$$F^* = \left\{ \begin{array}{lll} \langle\langle 01000\rangle, 1\rangle, & \langle\langle 10000\rangle, 0\rangle, & \langle\langle 00001\rangle, 0\rangle, \\ \langle\langle 00010\rangle, 0\rangle, & \langle\langle 00100\rangle, 1\rangle, & \langle\langle 10001\rangle, 0\rangle, \\ \langle\langle 00011\rangle, 0\rangle, & \langle\langle 00110\rangle, 1\rangle, & \langle\langle 01100\rangle, 0\rangle, \\ \langle\langle 11000\rangle, 1\rangle & & \end{array} \right\}.$$

The table placed below includes elements from the set F^* connected into pairs in which the sequences L_{k_p} differ only in one place.

1	2	3	4	5
$\langle\langle 01000\rangle, 1\rangle$ $\langle\langle 11000\rangle, 1\rangle$	$\langle\langle 10000\rangle, 0\rangle$ $\langle\langle 11000\rangle, 1\rangle$	$\langle\langle 01000\rangle, 1\rangle$ $\langle\langle 01100\rangle, 0\rangle$	$\langle\langle 00001\rangle, 0\rangle$ $\langle\langle 00011\rangle, 0\rangle$	$\langle\langle 10000\rangle, 0\rangle$ $\langle\langle 10001\rangle, 0\rangle$
$\langle\langle 00001\rangle, 0\rangle$ $\langle\langle 10001\rangle, 0\rangle$	$\langle\langle 00100\rangle, 1\rangle$ $\langle\langle 01100\rangle, 0\rangle$	$\langle\langle 00010\rangle, 0\rangle$ $\langle\langle 00110\rangle, 1\rangle$	$\langle\langle 00100\rangle, 1\rangle$ $\langle\langle 00110\rangle, 1\rangle$	$\langle\langle 00010\rangle, 0\rangle$ $\langle\langle 00011\rangle, 0\rangle$
0	1	1	0	0

It should be noted that 2^{nd} and 3^{rd} columns of the table include the pairs $F_j = \langle L_j, c_j\rangle$ and $F_k = \langle L_k, c_k\rangle$ for which $c_j \neq c_k$. According to theorem (3.1) neighbourhood N_Z gets nonzero value for elements with indices 2 and 3 which means that $N_Z = \langle 01100\rangle$. Hence

$$F^{*|N_Z} = \{\langle\langle 00\rangle, 0\rangle, \langle\langle 01\rangle, 1\rangle, \langle\langle 10\rangle, 1\rangle, \langle\langle 11\rangle, 0\rangle\}.$$

4 The definition of a learning process of the CELLS system

CELLS is a type of the learning system with the supervision. The main step in the construction of the CELLS system is identifying the training information in it with the set F^*.

Definition 4.1 A system able to perform minimization on the training information set F^* will be called the learning system CELLS.

Minimization in the CELLS system will be performed on the basis of the theorem about the neighbourhood.

5 Examples of appications of the CELLS system

5.1 Example 1

Figure 1 shows a product of some cellular automaton evolution from random initial configuration. The aim of the CELLS system was to find local rules determining the evolution CA. Additional difficulty consists in including areas with false information into the product CA.

The CELLS system recognizes an image as a product of evolution of two-states evolution CA ($C = \{0, 1\}$) for rules

$$\begin{aligned} N_Z &= \langle 1, 1, 0, 0, \ldots, 0 \rangle, \\ g_{CA} &= \{\langle\langle 0, 0\rangle, 0\rangle, \langle\langle 0, 1\rangle, 1\rangle, \langle\langle 1, 0\rangle, 1\rangle, \langle\langle 1, 1\rangle, 0\rangle\} \end{aligned} .$$

Subsequently CELLS marked areas which had not arised as a result of applying that rule. This is shown in Fig. 2.

5.2 Example 2

Similar rule to a rule shown in example 1 was used to finding early pathological changes in human tissue with the use of mammography pictures analysis. It was assumed that the healthy tissue has a construction which can be described by the evolution rules of CA with constant ϵ of deviation from those rules. Areas with the value ϵ higher than the mean one were indicated by the system as areas with pathology (Fig. 3 and 4).

6 Conclusion

There are a lot of physical, chemical and social phenomena for which successful tests of their simulating could be presented using cellular automata. Therefore, it is valid to create a system which will be able to find rules of the phenomenon evolution on the basis of the analysis of the product CA. Such a learning system can be useful in the analysis of the examined phenomenon and in finding mechanisms ruling its behavior.

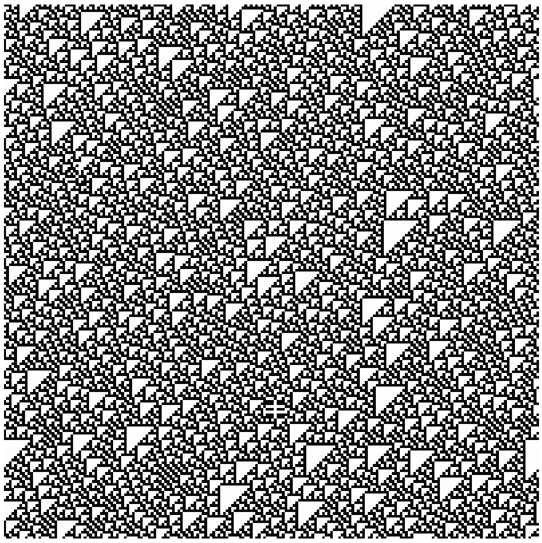

Fig. 1. *CA* product with false areas

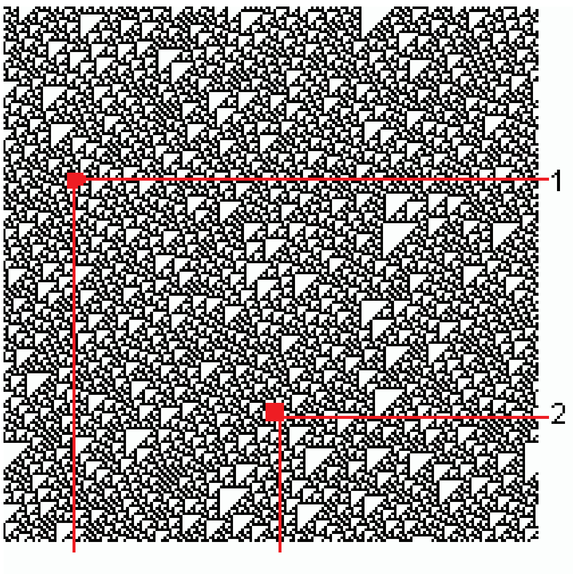

Fig. 2. *CA* product with false areas marked by CELLS

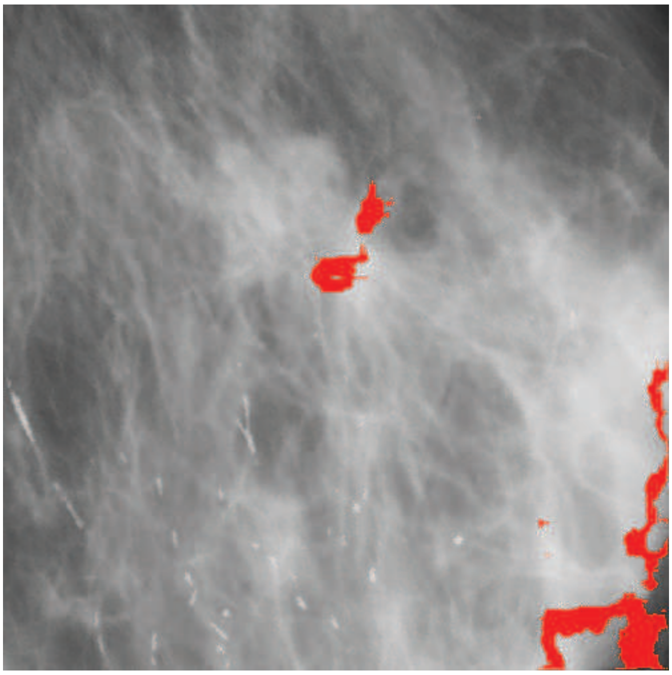

Fig. 3. Mammography picture with areas of pathology marked by CELLS

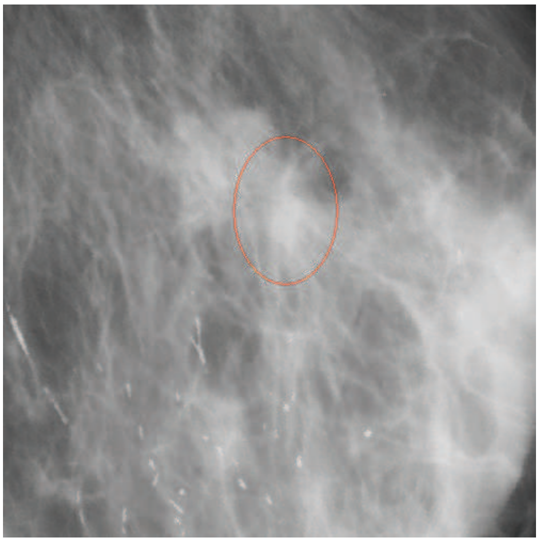

Fig. 4. Picture 3 with areas of pathology marked by medical specialist

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