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NOTION OF DISTANCE FOR EUCLIDEAN PLANE

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Abstract. One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The purpose is to choose a certain fundamental set of properties from which the other properties of the object can be deduced. Some of axioms of Euclidean plane based on the notion of distance are considered. The notions of linear and planar sets are introduced in terms of distance. Thus Euclidean plane is regarded as a distance space with a metric satisfying the corresponding properties.

1. Introduction

Axiomatic geometry made its debut with the Greeks in the sixth century BC, who insisted that statements be derived by logic and reasoning rather than trial and error. This systematization exteriorized in the thirteen volume *Elements* by Euclid (300 BC). Euclid's geometry prevailed until the 19th century when the discovery of non-Euclidean geometry. Several great mathematicians including Pasch, Peano, Pieri, Veblen, Levi and Hilbert have made splendid improvements in Euclidean geometry as a mathematical (axiomatic) system. Hilbert partitioned his axioms for Euclidean geometry into five groups: connection, order, congruence, parallels, continuity.

We describe some properties of various sets for the usual Euclidean 2–space in terms of distance (or metric) as the only initial notion.

2. Metric space

Definition 2.1. Let S be an arbitrary nonempty set. A function $d : S\times S \to \mathbb{R}$ is a *distance function* or *metric* on S if, and only if, for each $p, q, r \in S$

- $d(p,q) \geq 0$, and $d(p,q) = 0$ iff $p = q$ (POSITIVE PROPERTY);
- $d(p,q) = d(q,p)$ (SYMMETRIC PROPERTY);
- $d(p,q) \leq d(p,r) + d(r,q)$ (TRIANGLE INEQUALITY).

We call the set S endowed with this metric a *metric space* and, for all $p,q \in S$, we call the number $d(p,q)$ the *distance* between p and q (with respect to the metric d).

Example 2.1. Define d on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows: $d(p,p) = 0$ for all $p \in \mathbb{R}^2$ and for $p,q \in \mathbb{R}^2$ with $p \neq q$, $d(p,q) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ if neither p nor q is the origin $(0,0)$ of $\mathbb{R}^2$ and $d(p,q) = 1 + \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ otherwise. Then d has the positive property and is symmetric. Moreover, d *coincident* with the usual Euclidean metric on $\mathbb{R}^2$ except when exactly one of p and q is the origin. For all $p,q,r \in \mathbb{R}^2$, we have $d(p,(0,0)) \leq d(p,r) + d(r,(0,0))$ and $d(p,q) \leq d(p,(0,0)) + d((0,0,q))$, by the triangle inequality for the Euclidean metric on $\mathbb{R}^2$, so that d also satisfies the triangle inequality and it is consequently a metric on $\mathbb{R}^2$.

Definition 2.2. Suppose (S,d) and (T,e) are metric spaces and $f : S \longrightarrow T$. Then f is called an *isometry* or an *isometric map* if, and only if, $e(f(p),f(q)) = d(p,q)$ for all $p,q \in S$. If f is an isometry, we say that the metric subspace $(f(S),e)$ of (T,e) is an *isometric copy* of the space (S,d).

3. Sets in metric space

Metrics are designed to measure distance between points. Now, we consider the notion of linear set in an arbitrary metric space (see [2]).

Definition 3.1. Let A, B, and C be three points in a metric space (S,d). We say that $\{A,B,C\}$ is a *linear set* in S if

$$d(A,B) = \pm d(A,C) \pm d(C,B). \tag{LS}$$

If both summands on the right-hand side are not zero and have sign plus, then we say that C lies *between* A and B, and we write $A - C - B$.

Definition 3.2. The set X in a metric space (S,d) is called *linear set* if each three different points in X form a linear set.

Definition 3.3. The *segment AB with endpoints A and B* is the set $\overline{AB} = \{A,B\} \cup \{C : A - C - B\}$. The *distance* $d(A,B)$ from A to B is the length $|AB|$ of $\overline{AB}$. The union $\overleftrightarrow{AB}$ of all segments containing $\overline{AB}$ is the (straight) *line* through A and B.

Definition 3.4. Suppose A and B are points in a metric space (S, d). The *ray* AB outgoing from A towards B is the set $\overrightarrow{AB}$ consisting of $\overline{AB}$ and all points C with $|AC| = |AB| + |BC|$. The ray of line AB emanating from A in the direction opposite to B is the set $\overleftarrow{AB}$ of all points C such that $|CB| = |CA| + |AB|$.

The concept of linear sets in a metric space without additional conditions is not interesting. For example, a line is not necessarily a linear set, as the next example shows.

Example 3.1. Let $S = \{A, B, C, D\}$, $d(A, B) = 6$, $d(A, C) = 4$, $d(A, D) = d(B, D) = d(C, D) = 3$ and $d(B, C) = 2$. It is easy to see that $S = \overline{AB}$ complies with the axioms of metric space. On the other hand, the set $\{A, C, D\}$ is not linear, hence the segment $\overline{AB}$ is not linear. We also see that three points can lie on a line which is not a linear set, whenever a line is not the union of two rays (half-lines) emanating from a point in opposite directions. To show this, consider the line $BC = \overline{AB}$. We have $\overrightarrow{BC} = \{A, B, C\}$ and $\overleftarrow{BC} = \{B, C\}$. It follows that $BC \neq \overrightarrow{BC} \cup \overleftarrow{BC}$.

Remark 3.1. If a finite set S is linear, then there exists a segment containing S with endpoints from S. Really, among all pairs belonging to S, there exists a pair A, B with maximal distance. Let C be another point of S. Then $\{A, B, C\}$ is a linear set. From the definition of the betweenness follows that $|AB| = |AC| + |CB|$, and thus $C \in \overline{AB}$.

Note, the linearity of a metric space (S, d) does not imply the uniqueness of point C (with respect to the points $A, B \in S$) satisfying the equation (LS). Indeed, suppose (S, d) is metric space, $S = \{K, L, M, N\}$ and $d(K, L) = d(L, M) = d(M, N) = d(N, K) = 1$, $d(K, M) = d(L, N) = 2$. Each three points of S form a linear set in which one of the distances is 2 and the others are 1. In this case, K and M lie between L and N, and we have $d(K, L) = d(L, M)$ and $d(M, N) = d(N, K)$.

Next, we consider the notion of linearly coupled sets and the notion of a triangle in an arbitrary metric space (see [2]).

Definition 3.5. Two sets are *linearly coupled* if their intersection contains at least two points.

Definition 3.6. A finite system of sets in (S, d) is *linearly coupled* if they can be numbered so that every two neighboring sets are linearly coupled.

Definition 3.7. Suppose A, B, C are points of the metric space (S, d). We say that they define a *triangle* ABC. The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ are the *sides* of the triangle, and A, B, C are the *vertices*. The number $|ABC| = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle, is the *area* of $\triangle ABC$. A triangle with zero area is said to be *degenerated*.

Definition 3.8. We say that a point M of the space (S, d) *belongs to a triangle* ABC if

$$|ABC| = |MAB| + |MBC| + |MCA|. \qquad (\triangle)$$

In this sense, any triangle is regarded as the set of points belonging to it.

From the definition of a triangle follows that if $\triangle ABC$ is degenerated, then $\{A, B, C\}$ is a linear set. In this case, one of the factors in $s(s-a)(s-b)(s-c)$ vanishes. For example, let $s - c = 0$, i.e., $c = a + b$ and we can assume (without loss of generality) $a = |CB|$, $b = |AC|$, $c = |AB|$. We obtain $|AB| = |AC| + |CB|$, i.e., $A - C - B$.

4. The metric axioms

Next, we introduce one of possible modifications of metric axioms, which hold true in Euclidean 2-space (cf. [2]).

M1. (UNIQUENESS AXIOM) For each two points A and B, a point M satisfying the condition $|AB| = \pm|AM| \pm |MB|$, where $|AM|$, $|MB|$ and the sign are fixed, is uniquely determined.

M2. (LINEARITY AXIOM) The union of sets of any linearly coupled system of linear sets is a linear set.

M3. (AXIOM OF FILLING A LINE) For each two points A and B and any relation $d(A, B) = \pm d_1 \pm d_2$, $d_1 \geq 0$, $d_2 \geq 0$, there exists a point C such that $d(A, C) = d_1$ and $d(C, B) = d_2$.

M4. There are three points not forming a linear set.

M5. For an arbitrary line l and any point C, we have $|ABC| = \lambda|AB|$, where λ does not depend on the choice of the segment $\overline{AB} \subset l$.

M6. (Axiom of similarity) Let $d(A, A') + d(A', O) = d(A, O)$ and $d(B, B') + d(B', O) = d(B, O)$, where all distances differ from zero, and let

$$\frac{d(A', O)}{d(A, O)} = \frac{d(B', O)}{d(B, O)}.$$

Then

$$\frac{d(A', O)}{d(A, O)} = \frac{d(B', O)}{d(B, O)} = \frac{d(A', B')}{d(A, B)}.$$

M7. (Axiom of congruence) Let $d(A, B) = d(A', B')$. Then there exists an isometry $f : (S, d) \longrightarrow (S, d)$ such that $f(A) = A'$ and $f(B) = B'$.

Now, let us observe some connections between this system of axioms and the notions of planar set presented in the previous sections. First of all we can see that if a metric space satisfies axiom **M2**, then its segments are the linear sets. Indeed, let $\overline{AB}$ be a segment in (S, d). Then $A \neq B$ since all one-element sets are linear. Suppose that K, L, M are points of $\overline{AB}$. By the definition of a segment, we have $\{A, B, K\}$, $\{A, B, L\}$ and $\{A, B, M\}$ are linearly coupled linear sets, and by **M2**, their union $\{A, B, K, L, M\}$ is a linear set. This means, that its subset $\{K, L, M\}$ is linear, too.

Looking at definitions 3.7 and 3.8 above, we see that each vertex of the triangle belongs to it. The situation with the sides of triangle is not trivial. To prove this, we need to consider the axiom **M5**. Suppose $\triangle ABC$ is a triangle in (S, d). If $M \in \overline{AB}$ then $|AB| = |AM| + |MB|$. If we multiply both sides of this equation by real number λ (λ is the *altitude* of the triangle $\triangle ABC$ or the distance from C to AB) we have $|ABC| = |AMC| + |MBC|$. Then $|ABC| = |AMB| + |AMC| + |MBC|$ since $|AMB| = 0$. From this and by the definition of a triangle, we conclude that M belongs to $\triangle ABC$.

We recall here only one important theorem related to lines of a metric space.

Theorem (A. A. Ivanov [2]). If a metric space (S, d) satisfies metric axioms **M1**-**M3**, then any line of this space is isometric to the real line with the usual metric.

The system of metric axioms for Euclidean 2-space presented above may be viewed as a system equivalent to the Hilbert's axiomatic. The connections between them seems to be an interesting subject for investigation.

References

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