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SOME PROPERTIES OF GENERALIZED TRIBONACCI QUATERNIONS

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Abstract

In this paper we introduce distinct types of Tribonacci quaternions. We describe dependences between them and we give some their properties also related to a matrix representation.

1. Introduction

Let $\mathbb{H}$ be the set of quaternions z of the form

$$z = a + bi + cj + dk, \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and i, j, k are complex operators such that

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

and

$$ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j. \tag{3}$$

If $z_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $z_2 = a_2 + b_2 i + c_2 j + d_2 k$ are any two quaternions then the equality, the addition, the substraction and the multiplication by scalar are defined as follows.

Equality: $z_1 = z_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\, i + (c_1 + c_2)\, j + (d_1 + d_2)\, k$,
substraction: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)\, i + (c_1 - c_2)\, j + (d_1 - d_2)\, k$,
multiplication by scalar $s \in \mathbb{R}$: $sz_1 = sa_1 + sb_1 i + sc_1 j + sd_1 k$.
The quaternion multiplication is defined using (2).
The conjugate of a quaternion z is defined by

$$z^* = (a + bi + cj + dk)^* = a - bi - cj - dk. \tag{4}$$

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Moreover we use the following notation for
real part: $\Re z = (z + z^*)/2 = a \in \mathbb{R}$,
imaginary part: $\Im z = (z - z^*)/2 = bi + cj + dk \in \mathbb{H}$.
The norm of a quaternion z is defined by

$$N(z) = a^2 + b^2 + c^2 + d^2. \tag{5}$$

For the basics on the quaternions theory, see [12].

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = F_1 = 1$. There are many numbers defined by linear recurrence relations and they are also named as numbers of the Fibonacci type. We list some of them.

- $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$ – Lucas numbers,
- $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$ – Pell numbers,
- $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with $Q_0 = 2$, $Q_1 = 2$ – Pell-Lucas numbers,
- $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$ – Jacobsthal numbers,
- $j_n = j_{n-1} + 2j_{n-2}$, for $n \geq 2$ with $j_0 = 2$, $j_1 = 1$ – Jacobsthal-Lucas numbers.

These numbers have many applications in distinct areas of mathematics also in the quaternions theory.

In 1963 Horadam [5] introduced nth Fibonacci and Lucas quaternions. Three decades later in [6] Horadam mentioned about the possibility of introducing Pell quaternions and generalized Pell quaternions. Interesting results concerning Pell quaternions, Pell-Lucas quaternions have been obtained quite recently and can be found in [2], [11]. Jacobsthal quaternions and Jacobsthal-Lucas quaternions were introduced in [10].

In the most recent paper of G. Cerda-Morales (see [1]) we can find the definition of generalized Tribonacci quaternions due to their coefficients. The definition of these quaternions is based on the definition of generalized Tribonacci numbers V_n

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},\ for\ n \geq 3,$$

where $V_0 = a$, $V_1 = b$, $V_2 = c$ are arbitrary integers and r, s, t are real numbers. For $r = s = t = 1$ we have the set of quaternions defined in this paper. We present some properties of generalized Tribonacci quaternions, in particular relations between them.

2. The Tribonacci numbers

Let $n \geq 0$ be integer. The nth Tribonacci number T_n is defined by $T_0 = 1$, $T_1 = 1$, $T_2 = 2$, and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ for } n \geq 3. \tag{6}$$

Tribonacci numbers have been firstly defined by Feinberg in 1963, see [3].

The characteristic equation of (6) has the form $x^3 - x^2 - x - 1 = 0$ and it has roots

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\beta = \frac{1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}}{3},$$
$$\gamma = \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where

$$\omega = \frac{-1 + \epsilon\sqrt{3}}{2}, \ \epsilon^2 = -1.$$

Hence the Binet formula for the Tribonacci number T_n has the form

$$T_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)}. \tag{7}$$

There are some versions of Tribonacci numbers defined by the same linear recurrence relation as T_n but with different initial conditions.

The nth generalized Tribonacci number t_n is a number defined recursively by the recurrence relation of the form $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for $n \geq 3$ with fixed t_0, t_1, t_2. For special value of t_0, t_1, t_2 we obtain different kinds of Tribonacci numbers. If $t_0 = 1$, $t_1 = 1$, $t_2 = 2$ then we obtain the definition of T_n. Apart Tribonacci numbers T_n we define other kinds of Tribonacci numbers namely numbers R_n, S_n and U_n. For $n \geq 0$ we define three types of Tribonacci numbers as follows

$R_0 = 3,\ R_1 = 1,\ R_2 = 3$ and $R_n = R_{n-1} + R_{n-2} + R_{n-3}$ for $n \geq 3$,

$S_0 = 3,\ S_1 = 2,\ S_2 = 5$ and $S_n = S_{n-1} + S_{n-2} + S_{n-3}$ for $n \geq 3$,

$U_0 = 0,\ U_1 = 1,\ U_2 = 2$ and $U_n = U_{n-1} + U_{n-2} + U_{n-3}$ for $n \geq 3$.

The Table 1 presents values of these Tribonacci numbers for $n = 0, 1, ..., 10$.

n	0	1	2	3	4	5	6	7	8	9	10
T_n	1	1	2	4	7	13	24	44	81	149	274
R_n	3	1	3	7	11	21	39	71	131	241	443
S_n	3	2	5	10	17	32	59	108	199	366	673
U_n	0	1	2	3	6	11	20	37	68	125	230

Table 1.

Above Tribonacci numbers were considered in [3], [7], [8], [9] where among other Binet formulas for them were found. Moreover in [8] some relations between Tribonacci numbers were given. We recall these dependences

$$R_n = T_{n-1} + 2T_{n-2} + 3T_{n-3},\ for\ n \geq 3 \tag{8}$$

$$S_n = 3T_n - T_{n-1},\ for\ n \geq 1 \tag{9}$$

$$U_n = T_{n-1} + T_{n-2},\ for\ n \geq 2 \tag{10}$$

$$\sum_{l=0}^{n} U_l = T_{n+1} - 1 \tag{11}$$

$$\sum_{l=1}^{n} R_l = 2U_{n+1} + U_{n-1} - 3 \tag{12}$$

$$\sum_{l=0}^{n} S_l = \frac{3U_{n+2} + 2U_{n+1} - U_n - 2}{2} \tag{13}$$

$$\sum_{l=0}^{n} T_l = \frac{U_{n+2} + U_{n+1} - 1}{2}. \tag{14}$$

From the above identities we can obtain other relations

$$2T_n = U_{n+1} + U_{n-1},\ for\ n \geq 1 \tag{15}$$

$$\sum_{l=1}^{n} R_l = 3T_n + T_{n-1} - 3 \tag{16}$$

$$\sum_{l=0}^{n} S_l = T_{n+2} + 2T_n - 1 \tag{17}$$

$$\sum_{l=0}^{n} T_l = \frac{T_{n+2} + T_n - 1}{2}. \tag{18}$$

3. The Tribonacci quaternions

For $n \geq 0$ the nth generalized Tribonacci quaternion TQ_n is defined as

$$TQ_n = t_n + it_{n+1} + jt_{n+2} + kt_{n+3}. \tag{19}$$

In particular for Tribonacci numbers we obtain distinct Tribonacci quaternions. Using presented earlier Tribonacci numbers T_n, S_n, R_n and U_n we have four types of Tribonacci quaternions. Then

$$TTQ_n = T_n + iT_{n+1} + jT_{n+2} + kT_{n+3} \tag{20}$$

$$TRQ_n = R_n + iR_{n+1} + jR_{n+2} + kR_{n+3} \tag{21}$$

$$TSQ_n = S_n + iS_{n+1} + jS_{n+2} + kS_{n+3} \tag{22}$$

$$TUQ_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3} \tag{23}$$

Firstly we give relations between Tribonacci quaternions.

Theorem 1. *Let n be an integer. Then*

(i) $TRQ_n = TTQ_{n-1} + 2TTQ_{n-2} + 3TTQ_{n-3},\ for\ n \geq 3,$

(ii) $TSQ_n = 3TTQ_n - TTQ_{n-1},\ for\ n \geq 1,$

(iii) $TUQ_n = TTQ_{n-1} + TTQ_{n-2},\ for\ n \geq 2,$

(iv) $2TTQ_n = TUQ_{n+1} + TUQ_{n-1},\ for\ n \geq 1.$

Proof. (i) Using (21) and (8) we have

$$\begin{aligned} TRQ_n &= R_n + iR_{n+1} + jR_{n+2} + kR_{n+3} = \\ &= (T_{n-1} + 2T_{n-2} + 3T_{n-3}) + i\,(T_n + 2T_{n-1} + 3T_{n-2}) + \\ &\quad +j\,(T_{n+1} + 2T_n + 3T_{n-1}) + k\,(T_{n+2} + 2T_{n+1} + 3T_n) = \\ &= (T_{n-1} + iT_n + jT_{n+1} + kT_{n+2}) + \\ &\quad +2\,(T_{n-2} + iT_{n-1} + jT_n + kT_{n+1}) + \\ &\quad +3\,(T_{n-3} + iT_{n-2} + jT_{n-1} + kT_n) = \\ &= TTQ_{n-1} + 2TTQ_{n-2} + 3TTQ_{n-3}. \end{aligned}$$

In the same way, using (9), (10) and (15) one can easily prove identities (ii)-(iv). $\square$

The next theorem gives formulas for sums of Tribonacci quaternions.

Theorem 2. *Let n be an integer. Then*

(i) $\sum_{l=0}^{n} TUQ_l = TTQ_{n+1} - TTQ_0,$

(ii) $\sum_{l=1}^{n} TRQ_l = 2TUQ_{n+1} + TUQ_{n-1} - (3 + 4i + 7j + 14k)\,,$

(iii) $\sum_{l=0}^{n} TSQ_l = \dfrac{3TUQ_{n+2} + 2TUQ_{n+1} - TUQ_n}{2} - (1 + 4i + 6j + 11k)\,,$

(iv) $\sum\limits_{l=0}^{n} TTQ_l = \dfrac{TUQ_{n+2} + TUQ_{n+1} - (1+3i+5j+9k)}{2}$,

(v) $\sum\limits_{l=1}^{n} TRQ_l = 3TTQ_n + TTQ_{n-1} - (3+4i+7j+14k)$,

(vi) $\sum\limits_{l=0}^{n} TSQ_l = TTQ_{n+2} + 2TTQ_n - (1+4i+6j+11k)$,

(vii) $\sum\limits_{l=0}^{n} TTQ_l = \dfrac{TTQ_{n+2} + TTQ_n - (1+2i+4j+8k)}{2}$.

Proof. (i) Using (23) and (11) we have

$$\begin{aligned}
\sum_{l=0}^{n} TUQ_l &= TUQ_0 + TUQ_1 + \ldots + TUQ_n = \\
&= (U_0 + iU_1 + jU_2 + kU_3) + \\
&\quad + (U_1 + iU_2 + jU_3 + kU_4) + \ldots + \\
&\quad + (U_n + iU_{n+1} + jU_{n+2} + kU_{n+3}) = \\
&= (U_0 + U_1 + \ldots + U_n) + \\
&\quad + i\,(U_1 + U_2 + \ldots + U_{n+1}) + \\
&\quad + j\,(U_2 + U_3 + \ldots + U_{n+2}) + \\
&\quad + k\,(U_3 + U_4 + \ldots + U_{n+3}) = \\
&= T_{n+1} - 1 + i(T_{n+2} - 1 - U_0) + j(T_{n+3} - 1 - U_0 - U_1) + \\
&\quad + k\,(T_{n+4} - 1 - U_0 - U_1 - U_2) = \\
&= T_{n+1} + iT_{n+2} + jT_{n+3} + kT_{n+4} - (1 + i + 2j + 4k),
\end{aligned}$$

which ends the proof. In the same way one can easily prove (ii)-(vii). $\square$

4. The Binet formula and a matrix representation

Using the Binet formula for Tribonacci numbers T_n we can give the direct formula for nth Tribonacci quaternion

$$\begin{aligned}
TTQ_n &= \tfrac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \tfrac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \tfrac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)} + \\
&\quad + i\left(\tfrac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)} + \tfrac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)} + \tfrac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)}\right) + \\
&\quad + j\left(\tfrac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \tfrac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \tfrac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}\right) + \\
&\quad + k\left(\tfrac{\alpha^{n+5}}{(\alpha-\beta)(\alpha-\gamma)} + \tfrac{\beta^{n+5}}{(\beta-\alpha)(\beta-\gamma)} + \tfrac{\gamma^{n+5}}{(\gamma-\alpha)(\gamma-\beta)}\right).
\end{aligned}$$

For other types of Tribonacci quaternions we can obtain analogous formulas, we omit their presentations.

Matrix representations play an important role in the theory of numbers defined by the recurrence relations, see for example [4]. We give a matrix generator also for Tribonacci quaternions TQ_n.

Theorem 3. *Let*

$$T=\begin{bmatrix} -TQ_0-TQ_1+TQ_2 & TQ_1-TQ_0 & TQ_0 \\ TQ_0 & TQ_2-TQ_1 & TQ_1 \\ TQ_1 & TQ_0+TQ_1 & TQ_2 \end{bmatrix} \tag{24}$$

and

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{25}$$

Then

$$TA^n=\begin{bmatrix} TQ_{n-1} & TQ_{n-2}+TQ_{n-1} & TQ_n \\ TQ_n & TQ_{n-1}+TQ_n & TQ_{n+1} \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n+2} \end{bmatrix} \; for\ n\geq 2. \tag{26}$$

Proof. (by induction on n) If $n=2$ we have

$$A^2=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

and

$$TA^2=\begin{bmatrix} -TQ_0-TQ_1+TQ_2 & TQ_1-TQ_0 & TQ_0 \\ TQ_0 & TQ_2-TQ_1 & TQ_1 \\ TQ_1 & TQ_0+TQ_1 & TQ_2 \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}=$$

$$=\begin{bmatrix} TQ_1 & TQ_0+TQ_1 & TQ_2 \\ TQ_2 & TQ_1+TQ_2 & TQ_0+TQ_1+TQ_2 \\ TQ_0+TQ_1+TQ_2 & TQ_0+TQ_1+2TQ_2 & TQ_0+2TQ_1+2TQ_2 \end{bmatrix}=$$

$$=\begin{bmatrix} TQ_1 & TQ_0+TQ_1 & TQ_2 \\ TQ_2 & TQ_1+TQ_2 & TQ_3 \\ TQ_3 & TQ_2+TQ_3 & TQ_4 \end{bmatrix}.$$

Assume that

$$TA^n=\begin{bmatrix} TQ_{n-1} & TQ_{n-2}+TQ_{n-1} & TQ_n \\ TQ_n & TQ_{n-1}+TQ_n & TQ_{n+1} \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n+2} \end{bmatrix}.$$

We shall show that

$$TA^{n+1}=\begin{bmatrix} TQ_n & TQ_{n-1}+TQ_n & TQ_{n+1} \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n+2} \\ TQ_{n+2} & TQ_{n+1}+TQ_{n+2} & TQ_{n+3} \end{bmatrix}.$$

Using induction's hypothesis we have

$$TA^{n+1} = TA^nA = \begin{bmatrix} TQ_{n-1} & TQ_{n-2}+TQ_{n-1} & TQ_n \\ TQ_n & TQ_{n-1}+TQ_n & TQ_{n+1} \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n+2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} TQ_n & TQ_{n-1}+TQ_n & TQ_{n-2}+TQ_{n-1}+TQ_n \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n-1}+TQ_n+TQ_{n+1} \\ TQ_{n+2} & TQ_{n+1}+TQ_{n+2} & TQ_n+TQ_{n+1}+TQ_{n+2} \end{bmatrix} =$$

$$= \begin{bmatrix} TQ_n & TQ_{n-1}+TQ_n & TQ_{n+1} \\ TQ_{n+1} & TQ_n+TQ_{n+1} & TQ_{n+2} \\ TQ_{n+2} & TQ_{n+1}+TQ_{n+2} & TQ_{n+3} \end{bmatrix},$$

which ends the proof. □

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