

ON MONOTONICITY OF REAL FUNCTIONS

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ABSTRACT

Monotonicity of functions were of great interest of many mathematicians. Starting from the well known theorem of monotonicity of a differentiable function one can get quite sophisticated results. We give a survey of results when thesis of them is *continuous and monotone function*. Someone can ask why it should be continuous. Even a differentiable functions but not at the only point of its domain with positive derivative need not be non-decreasing. That is why we want to look for theorems for continuous functions.

WELL KNOWN THEOREMS ON MONOTONICITY

Starting his mathematical way any student knows that:

Theorem 1¹. *If $f : (a, b) \rightarrow \mathbb{R}$ fulfils the following conditions:*

- *f' exists at every point of (a, b) ,*
- *$f'(x) \geq 0$ if $x \in (a, b)$,*

then f is non-decreasing (and continuous).

It is not difficult to find out that:

If a continuous function $f : (a, b) \rightarrow \mathbb{R}$ is non-decreasing, then it is not necessary for f to be differentiable everywhere.

We shall try to give some weak conditions for a function to be continuous and non-decreasing.

Let us remind that a monotone function (say non-decreasing) must have a non-negative derivative in a big set of points.

Theorem 1. *If $f : (a, b) \rightarrow \mathbb{R}$ is non-decreasing function, then f' exists almost everywhere in (a, b) and $f'(x) \geq 0$.*

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¹I know nobody, who knows the author of this theorem.

It means that condition \mathcal{M} is fulfilled almost everywhere in the set E if the set of points at which this condition is not fulfilled has Lebesgue measure 0.

The converse statement is not true! There are monotone functions which are differentiable only in the set which complement has measure 0. For example:

Example 1.

Let K be the classical Cantor ternary set. Denote by (a_n, b_n) all components of the complement of the set K and define the function f as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in \{a_n : n \in \mathbb{N}\}, \\ 1 & \text{if } x \in \{b_n : n \in \mathbb{N}\}, \\ \text{linear and continuous} & \text{in each interval } [a_n, b_n]. \end{cases}$$

Let us discuss some generalizations of those theorem. There are different ways of generalizations:

- (1) $f'(x) \geq 0$ not everywhere,
- (2) instead of usual derivative one can consider generalized derivative,
- (3) both of them.

1. GENERALIZATIONS OF THE FIRST KIND

First series of theorems deals with ordinary derivatives. Their generalizations concern to the smaller set of points where the derivative exists.

Theorem 2. (G. Goldowsky – [2], 1928, L. Tonelli – [12], 1930)

If a function $f : (a, b) \rightarrow \mathbb{R}$ is continuous, f' exists nearly everywhere in (a, b) and $f'(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing.

The term *nearly everywhere* means that such property holds for the set which complement is at most countable.

Next theorem needs the idea of absolute continuity of a function. Let us remind this notion. We say that a function $f : (a, b) \rightarrow \mathbb{R}$ is absolutely continuous if for each positive ε there is a positive δ such that

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \varepsilon$$

for each sequence of intervals (a_k, b_k) for which

$$\sum_{k=1}^{\infty} |b_k - a_k| < \delta.$$

Theorem 3. (S. Saks – [7], 1937)

If a function $f : (a, b) \rightarrow \mathbb{R}$ is absolutely continuous and $f'(x) \geq 0$ almost everywhere, then f is non-decreasing.

When looking at the previous theorems we can observe that continuity of a function is contained in assumptions. Z. Zahorski proved interesting theorem for functions which were from the I class of Baire with Darboux property. But still the thesis says that such a function must be continuous.

Theorem 4. (Z. Zahorski – [13], 1950) *If a function $f : (a, b) \rightarrow \mathbb{R}$ belongs to the I class of Baire, is a Darboux function, f' exists nearly everywhere in (a, b) and $f'(x) \geq 0$ almost everywhere, then f is non-decreasing (and continuous).*

It is worth to add that none of the assumptions can be omitted. Here we give some counterexamples for it.

Example 2. Darboux property.

The function $f : [-1, 1] \rightarrow \mathbb{R}$ defined in a way

$$f(x) = \begin{cases} 1+x & \text{if } x \in [-1, 0), \\ x & \text{if } x \in [0, 1]. \end{cases}$$

has all Zahorski's assumptions but Darboux property.

Example 3. Existence of a derivative.

The function $f : [0, 1] \rightarrow \mathbb{R}$ defined in a way

$$f(x) = \begin{cases} 0 & \text{if } x \in \{a_n : n \in \mathbb{N}\}, \\ 1 & \text{if } x \in \{b_n : n \in \mathbb{N}\}, \\ \frac{1}{b_n - a_n} \cdot x - \frac{a_n}{b_n - a_n} & \text{if } x \in (a_n, b_n), \\ 0 & \text{if } x \in K \setminus (\{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}), \end{cases}$$

where (a_n, b_n) are all complement of the Cantor set K , has Darboux property, belongs to the first class of Baire and $f'(x) > 0$ for each x from the set $\bigcup_{n=1}^{\infty} (a_n, b_n)$. This function is differentiable at no point of the (uncountable) Cantor set.

2. GENERALIZATION OF THE SECOND KIND

2.1. Approximate derivative. First attempts to generalize theorems on monotonicity were given by Tolstoff in 1939. He applied approximate derivative instead of usual derivative, but he assumed a bit more than Zahorski did, it means that Tolstoff assumed that a function should be approximately continuous.

Theorem 5. (G. Tolstoff – [10], 1939) *If a function $f : (a, b) \rightarrow \mathbb{R}$ is approximately continuous, f'_{ap} exists nearly everywhere in (a, b) and $f'_{ap}(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing (and continuous).*

Z. Zahorski asked then: Is the next statement true?

Zahorski's hypothesis

If a function $f : (a, b) \rightarrow \mathbb{R}$ belongs to the I class of Baire, is a Darboux function, f'_{ap} exists nearly everywhere in (a, b) and $f'_{ap}(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing.

Positive answer has been given independently by A. M. Bruckner and T. Świątkowski in 1966. Then the proper theorem should be named Bruckner-Świątkowski Theorem.

Theorem 6. (T. Świątkowski – [8], 1966, A. M. Bruckner – [1], 1966)

If a function $f : (a, b) \rightarrow \mathbb{R}$

- (1) *belongs to the I class of Baire,*
- (2) *fulfils Darboux condition,*
- (3) *f'_{ap} exists nearly everywhere in (a, b) ,*
- (4) *$f'_{ap}(x) \geq 0$ almost everywhere,*

then f is non-decreasing (and continuous).

The proof of this theorem given by Świątkowski is as simple as can be. He used only fundamental properties of Darboux functions and approximate derivative and used no special tools of real functions theory. In spite of this proof, Bruckner involved several properties of real functions for example: Banach condition, VB and VBG functions, but the statement is the same. I am not able to say which of those proofs is better or more general. Both of them are brilliant and both of them are pretty long and complicated.

Approximate continuity can be regarded as continuity with respect to density topology. Thus approximate derivative is also the limit of appropriate quotient with respect to topology stronger than the natural one.

Thus we can come to next series of theorems.

2.2. Qualitative derivative.

Definition 1. *If $f : (a, b) \rightarrow \mathbb{R}$ is any function and $x_0 \in (a, b)$, then g is called the qualitative limit of f at the point x_0 if there exists a residual subset E of (a, b) such that*

$$\lim_{x \rightarrow x_0} f|_E(x) = g.$$

If we apply limit of this kind to differential quotient of a function, we get the idea of qualitative derivative of this function. Using this kind of

generalization of the usual derivative one can get the next theorem in our series.

Theorem 7. (J. L. Leonard – [4], 1972)

If a function $f : (a, b) \rightarrow \mathbb{R}$ fulfils Darboux condition, belongs to the I class of Baire, f'_q exists nearly everywhere in (a, b) and $f'_q(x) \geq 0$ almost everywhere, then f is non-decreasing (and continuous).

One can observe that this kind of a limit can be obtained as limit with respect to some topology. This topology is not a generalization of density topology, but it also is stronger than the natural one. This topology can be defined in the following way:

Definition 2. *A subset U of \mathbb{R} is called qualitatively open if it can be represented in the form*

$$U = G \triangle E,$$

where G is open in natural topology and E is of the first category.

There are several different topologies defined in a similar way. For example: let \mathcal{J} be a σ -ideal of subsets of \mathbb{R} . If a set U is regarded as open set in the topology generated by \mathcal{J} if it is a symmetrical difference of open set in natural topology and a set from the ideal \mathcal{J} , then we obtain a topology in \mathbb{R} .

Applying this topology to the idea of a limit of a function and to the differential quotient one can get quite a big class of theorems like the last one.

Generalizations of this idea will be found in further part of article.

3. GENERALIZATIONS OF THE THIRD KIND

3.1. Preponderant derivative.

Definition 3. *A number g is called a preponderant limit of a function $f : (a, b) \rightarrow \mathbb{R}$ at a point x_0 if there exists a measurable set E such that*

$$\liminf_{h \rightarrow 0^-} \frac{\mu(E \cap (x_0 - h, x_0))}{h} > \frac{1}{2}$$

$$\liminf_{h \rightarrow 0^+} \frac{\mu(E \cap (x_0, x_0 + h))}{h} > \frac{1}{2}$$

and

$$\lim_{x \rightarrow x_0} f|_E(x) = g.$$

Theorem 8. (J. Jędrzejewski – master's thesis, 1969)

If a function $f : (a, b) \rightarrow \mathbb{R}$ is from the I class of Baire, fulfils Darboux condition, f'_{pr} exists nearly everywhere in (a, b) and $f'_{pr}(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing (and continuous).

Theorem 9. (J. L. Leonard – [4], 1972)

If a function $f : (a, b) \rightarrow \mathbb{R}$ is preponderantly continuous, f'_{pr} exists nearly everywhere in (a, b) and $f'_{pr}(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing (and continuous).

The sets applied for preponderant limit do not generate any topology, then this kind of generalization is quite different from the previously considered manners.

3.2. Selective derivative.

Definition 4. (R. J. O'Malley – [5], 1977)

By a selection we mean a real function p of two variables which associates to each pair of points x and y a point $p(x, y)$ fulfilling the following conditions:

- (1) $p(x, y) = p(y, x)$ for each x and y from \mathbb{R} ,
- (2) if $x < y$, then $x < p(x, y) < y$.

Definition 5. We say that a number g is a selective limit with respect to the selection p of a function f at a point x_0 if

$$g = \lim_{y \rightarrow x_0} p(x_0, y).$$

Selective limit operation applied to differential quotient gives s -derivative.

Definition 6. A number $f'_s(x_0)$ is called selective derivative of a function f at the point x_0 if

$$f'_s(x_0) = \lim_{x \rightarrow x_0} \frac{f(p(x, x_0)) - f(x_0)}{p(x, x_0) - x_0}$$

for a given selection p .

Theorem 10. (R. J. O'Malley – [5], 1977)

If a function $f : (a, b) \rightarrow \mathbb{R}$ belongs to the I class of Baire, fulfils Darboux condition, f'_s exists nearly everywhere in (a, b) and $f'_s(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing (and continuous).

3.3. Świątkowski's τ -derivative.

Definition 7. (T. Świątkowski – [9], 1972)

For an $x \in \mathbb{R}$ let τ_x be a class of sets fulfilling the following conditions:

- (1) if $A \in \tau_x$ and $B \in \tau_x$ then $A \cap B \in \tau_x$,
- (2) if δ is a positive number and $E \in \tau_x$, then $E \cap (x - \delta, x + \delta) \in \tau_x$,
- (3) $\bigcap \tau_x = \{x\}$,
- (4) if $\delta > 0$ and $E \in \tau_x$, then $(E \cap (x - \delta, x + \delta)) \setminus \{x\} \neq \emptyset$.

Definition 8. We say that a class $\{\tau_x : x \in \mathbb{R}\}$ of sets fulfilling the above conditions fulfils Khintchine's condition if x_0 is τ -accumulation point of the set

$$\bigcup_{n=1}^{\infty} (x_n - \delta_n, x_n + \delta_n)$$

for every sequences $(x_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ such that

- $\lim_{n \rightarrow \infty} x_n = x_0$,
- $\lim_{n \rightarrow \infty} \delta_n = 0$, and $\delta_n > 0$, for each $n \in \mathbb{N}$,
- $\lim_{n \rightarrow \infty} \frac{\delta_n}{|x_n - x_0|} > 0$

Any class τ_x of subsets of the set of real numbers has some properties of a system of neighbourhoods of the point x . Applying the topological terminology, we say that a point x is called to be a τ -accumulation point of a set E if

$$E \cup \{x\} \in \tau_x.$$

Applying this τ -limit operation to the differential quotient of a function we obtain τ -derivative.

Theorem 11. (T. Świątkowski – [9], 1972)

A class of sets $\{\tau_x : x \in \mathbb{R}\}$ fulfils the Khintchine condition if and only if for each monotone function f from the existence of the τ -derivative of f implies the existence of f' .

Before we come to theorem on monotonicity, we have to define also condition (W).

Definition 9. We say that a function f and a class $\tau = \{\tau_x : x \in (a, b)\}$ satisfy condition (W) if

- (1) f fulfils Darboux condition,
- (2) f is nearly everywhere continuous in (a, b) ,
- (3) τ fulfils Khintchine's condition nearly everywhere in (a, b) ,
- (4) f' exists nearly everywhere in (a, b) .

Theorem 12. (M. Mastalerz-Wawrzyńczak – [6], 1977)

Let a class of sets $\tau = \{\tau_x : x \in \mathbb{R}\}$ fulfils conditions of Definition 7 and the Khintchine condition. Assume moreover that a function $f : (a, b) \rightarrow \mathbb{R}$ fulfils condition (W) with the class τ .

Under those assumptions, if $f'_\tau(x) \geq 0$ almost everywhere in (a, b) , then f is non-decreasing (and continuous).

3.4. Local systems.

Definition 10. (B. S. Thomson – [11], 1985)

By a local system we mean a class \mathcal{S} consisting of non-empty collections $\mathcal{S}(x)$ for each real number x , fulfilling the following conditions:

- (1) $\{x\} \notin \mathcal{S}(x)$,
- (2) $E \in \mathcal{S}(x) \rightarrow x \in E$,
- (3) $E \in \mathcal{S}(x) \wedge F \supset E \rightarrow F \in \mathcal{S}(x)$,
- (4) $E \in \mathcal{S}(x) \wedge \delta > 0 \implies E \cap (x - \delta, x + \delta) \in \mathcal{S}(x)$.

A local system is called filtering at a point x if

$$E \cap F \in \mathcal{S}(x) \text{ whenever } E \in \mathcal{S}(x) \text{ and } F \in \mathcal{S}(x).$$

A local system is called filtering if it is filtering at each x in \mathbb{R} .

A local system is called bilateral if

$$E \cap (x - \delta, x) \neq \emptyset \text{ and } E \cap (x, x + \delta) \neq \emptyset$$

for each $x \in \mathbb{R}$, $E \in \mathcal{S}(x)$ and $\delta > 0$.

Definition 11. (B. S. Thomson)

A number g is called \mathcal{S} -limit of a function f at a point x if

$$f^{-1}(g - \varepsilon, g + \varepsilon) \cup \{x\} \in \mathcal{S}(x)$$

for each positive ε . We shall write then

$$g = (\mathcal{S}) \lim_{t \rightarrow x} f(t).$$

If we apply the \mathcal{S} -limit operation to the differential quotient of a function f at a point x , then we get \mathcal{S} -derivative of the function f at x .

One of monotonicity criterion (involving generalized derivatives) is given by B. S. Thomson:

Theorem 13. (B. S. Thomson)

Let \mathcal{S} be a bilateral and filtering system fulfilling

- intersection condition i.e. for each class of sets

$$\{E_x \in \mathcal{S}(x) : x \in \mathbb{R}\}$$

there is a positive function $\delta : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$E_x \cap E_y \cap [x, y] \neq \emptyset$$

whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$,

- variation condition i.e. $\psi(I) \leq V_I(\psi, \mathcal{S})$ whenever ψ is a non-negative, sub-additive interval function.

Then if a function $f : (a, b) \longrightarrow \mathbb{R}$ belongs to the I class of Baire, fulfils Darboux condition, f'_S exists nearly everywhere in (a, b) and $f'_S(x) \geq 0$ almost everywhere, then f is non-decreasing (and continuous).

3.5. \mathfrak{B} -systems. Let us start from defining \mathfrak{B} classes.

Definition 12. For each $x \in \mathbb{R}$ let \mathfrak{B}_x^+ be a class of non-empty sets fulfilling the following properties:

- (1) $B_1 \cup B_2 \in \mathfrak{B}_x^+ \iff (B_1 \in \mathfrak{B}_x^+ \vee B_2 \in \mathfrak{B}_x^+)$,
- (2) $B \cap (x, x+t) \in \mathfrak{B}_x^+$ for each $B \in \mathfrak{B}_x^+$ and $t > 0$.

For each $x \in \mathbb{R}$ let \mathfrak{B}_x^- be a class of non-empty sets fulfilling the following properties:

- (1) $B_1 \cup B_2 \in \mathfrak{B}_x^- \iff (B_1 \in \mathfrak{B}_x^- \vee B_2 \in \mathfrak{B}_x^-)$,
- (2) $B \cap (x, x+t) \in \mathfrak{B}_x^-$ for each $B \in \mathfrak{B}_x^-$ and $t > 0$.

Let $\mathfrak{B}_x = \mathfrak{B}_x^- \cup \mathfrak{B}_x^+$.

Definition 13. If f defined in some (a, b) is a real function, then a number (or $+\infty$ or $-\infty$) is called \mathfrak{B} -limit number of f at x_0 from (a, b) if

$$\{x \in (a, b) : f^{-1}(U_g)\} \in \mathfrak{B}_{x_0}$$

for any neighbourhood U_g of the point g .

Definition 14. If

$$\{x \in (a, b) : f^{-1}(U_g) \in \mathfrak{B}_{x_0}^-\}$$

for any neighbourhood U_g of the point g , then g is called the left \mathfrak{B} -limit number of a function f at a point x_0 .

Similarly we define right \mathfrak{B} -limit numbers of a function f at a point x_0 .

- By $L_{\mathfrak{B}}^+(f, x_0)$ we denote the set of right \mathfrak{B} -limit numbers of f at x_0 .
- By $L_{\mathfrak{B}}^-(f, x_0)$ we denote the set of left \mathfrak{B} -limit numbers of f at x_0 .
- By $L_{\mathfrak{B}}(f, x_0)$ we denote the set of all \mathfrak{B} -limit numbers of f at x_0 .

Then, as for usual limit numbers, one can state:

Theorem 14. For arbitrary real function f on the interval (a, b) and any x_0 from (a, b) the sets $L_{\mathfrak{B}}(f, x_0)$, $L_{\mathfrak{B}}^-(f, x_0)$ and $L_{\mathfrak{B}}^+(f, x_0)$ are non-empty, closed and

$$L_{\mathfrak{B}}(f, x_0) = L_{\mathfrak{B}}^-(f, x_0) \cup L_{\mathfrak{B}}^+(f, x_0).$$

Up to now we have defined \mathfrak{B} -limit numbers of a function, we shall apply rather \mathfrak{B} -limits instead limit numbers. Let us define them.

Definition 15. A number g is called \mathfrak{B} -limit of a function at a point x_0 from (a, b) if

$$\{g\} = L_{\mathfrak{B}}(f, x_0).$$

There is another possibility to characterized \mathfrak{B} -limits of a function. But before we do this we shall have to define the second class of sets denoted by \mathfrak{B}_x^* for all $x \in \mathbb{R}$.

Definition 16. A subset E of \mathbb{R} belongs to the class \mathfrak{B}_x^* if $\mathbb{R} \setminus E \notin \mathfrak{B}_x$.

Now we can give the following characterization of \mathfrak{B} -limit of a function.

Theorem 15. A number g is \mathfrak{B} -limit of a function at a point $x_0 \in (a, b)$ if and only if

$$\{x \in (a, b) : f^{-1}(U_g) \in \mathfrak{B}_{x_0}^*\}$$

for any neighbourhood U_g of the point g .

We know from previous theorems that any function has \mathfrak{B} -limit number at any point of domain of f . This time we are not able to state that any function has \mathfrak{B} -limit, as it is evident for usual limits, but if a \mathfrak{B} -limit of a function exists it must be only one.

Let us remark yet that the class \mathfrak{B}^* that is applied to our considerations is very similar to the class τ considered by T. Świątkowski.

Next properties will be of some use in the further theory.

Definition 17. We say that the class \mathfrak{B} fulfils condition \mathcal{M} if

$$\bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}_{x_0}$$

for any: $x_0 \in (a, b)$, sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 and every sequence of sets $(E_n)_{n=1}^{\infty}$ such that $E_n \in \mathfrak{B}_{x_n}$.

Definition 18. We say that the class \mathfrak{B} fulfils condition \mathcal{M}' if

$$\bigcup_{n=1}^{\infty} E_n \in \mathfrak{B}_{x_0}$$

for any: $x_0 \in (a, b)$, sequence $(x_n)_{n=1}^{\infty}$ converging to x_0 and every sequence of intervals $(E_n)_{n=1}^{\infty}$ such that $E_n \in \mathfrak{B}_{x_n}$.

Now we want to compare \mathfrak{B} -derivatives with usual derivatives for monotone and continuous functions. Of course, if a function is differentiable it must be \mathfrak{B} -differentiable.

Assume now that a system \mathfrak{B} fulfils condition \mathcal{M}' and an increasing and continuous function f is \mathfrak{B} -differentiable and at some point $x \in (a, b)$ is not differentiable, for example it is not right differentiable. Then there are two numbers α, β and sequences $(u_n)_{n=1}^{\infty}, (w_n)_{n=1}^{\infty}$ converging to x and such that

$$x < u_{n+1} < w_{n+1} < u_n < w_n,$$

$$\frac{f(u_n) - f(x)}{u_n - x} < \alpha < \beta < \frac{f(w_n) - f(x)}{w_n - x}$$

for each positive integer n .

Then there are non-empty intervals $(u_n, u_n + \gamma_n)$, $(w_n - \delta_n, w_n)$ such that

$$\frac{f(t) - f(x)}{t - x} < \alpha < \beta < \frac{f(s) - f(x)}{s - x}$$

for all $t \in (w_n - \delta_n, w_n)$ and $s \in (u_n, u_n + \gamma_n)$ and $n \in \mathbb{N}$. In view of condition \mathcal{M}' of the system \mathfrak{B} the conditions

$$\bigcup_{n=1}^{\infty} (w_n - \delta_n, w_n) \in \mathfrak{B}_x \quad \text{and} \quad \bigcup_{n=1}^{\infty} (u_n, u_n + \gamma_n) \in \mathfrak{B}_x.$$

hold. Then there are \mathfrak{B} -limit numbers of the function $\frac{f(y) - f(x)}{y - x}$ at x , one of them less than α and the second greater than β , what means that the function f is not \mathfrak{B} -differentiable at x .

In that way we have proved:

Theorem 16. *If a system \mathfrak{B} fulfils condition \mathcal{M}' then \mathfrak{B} -differentiability of a monotone function is equivalent to differentiability of that function.*

4. TOPOLOGICAL APPROACH

Let \mathcal{T}_0 be the natural topology in the set \mathbb{R} . Assume that \mathcal{T} is a topology in \mathbb{R} stronger than \mathcal{T}_0 and such that each x in \mathbb{R} is a bilateral \mathcal{T} -accumulation point of \mathbb{R} , then the class $\mathfrak{B} = \{\mathfrak{B}_x : x \in \mathbb{R}\}$ is a class fulfilling conditions of J. Jędrzejewski, (see Definition 12). and conditions of T. Świątkowski (Definition 7). Moreover it forms also local system of B. Thomson, not every local system but only filtering one. Since those ideas are derived from a topology, then it is logical to use topological terminology.

Let us observe that there are several topologies in \mathbb{R} generating system \mathfrak{B} which fulfils condition \mathcal{M}' and consequently \mathfrak{B} -derivative of an increasing function coincides with usual derivative.

Theorem 17. *If \mathcal{T} is a topology in \mathbb{R} fulfilling the above conditions and for any monotone function from the existence of \mathcal{T} -derivative implies the existence of the natural derivative, then each interval is a connected set (with respect to the topology \mathcal{T}).*

For the proof of this theorem see Theorem 16 or Theorem 12 in [6].

In the end, let us see that if a non-empty interval I or \mathbb{R} is a disconnected set in topology \mathcal{T} , then even continuity of a function does not force a function f fulfilling conditions (2) – (4) from Theorem 6.

Here a problem has arisen:

PROBLEM.

Suppose that \mathcal{T} is a topology in \mathbb{R} for which each interval is a connected set (with respect to \mathcal{T}). Let $f'_{\mathcal{T}}$ denotes the derivative of f with respect to \mathcal{T} . Is it true that each function f defined in an interval (a, b) (or in \mathbb{R}) fulfilling the following conditions:

- f is Baire class 1,
- f fulfils Darboux condition,
- $f'_{\mathcal{T}}$ exists nearly everywhere,
- $f'_{\mathcal{T}}(x) \geq 0$ almost everywhere,

is monotone?

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