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ON THE ALIENATION OF THE CAUCHY EQUATION AND THE LAGRANGE EQUATION

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Abstract

In this article we look for all solutions of the Cauchy-Lagrange functional equation. The idea of considering such an equation is associated with the alienation phenomenon.

1. Introduction

In 1988 Dhombres in his article prove following

Theorem 1 (Dhombres, see [1]). *Let X and Y be two unitary rings and X be 2-divisible. Then each solution $f : X \to Y$ of the equation*

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y), \tag{1}$$

where $x, y \in X$, such that $f(0) = 0$ yields a solution of the system

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \tag{2}$$

for $x, y \in X$.

Adding sidewise equations in the system (2), we obtain the equation (1). It turns out that (2) and (1) are equivalent if and only if $f(0) = 0$. The above effect is called *the alienation phenomenon*. This kind of results, as well as their various generalizations, were considered in [2]-[9] and [12]-[14].

Our studies are connected with

Theorem 2 (Aczél, 1963, see [11]). *Let functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the Lagrange functional equation*

$$g(x) - g(y) = (x - y)f\left(\frac{x+y}{2}\right)$$

for all $x, y \in \mathbb{R}$. *Then, there exist constants* α, β, γ *such that*

$$\begin{cases} g(x) = \alpha x^2 + \beta x + \gamma \\ f(x) = 2\alpha x + \beta \end{cases}$$

for $x \in \mathbb{R}$.

We present the main result of the article [14] associated with the above theorem.

Theorem 3. *Let* $(R, +, \cdot)$ *be a uniquely 2-divisible ring. Then each solution* $f, g : R \to R$ *of the functional equation*

$$f(x+y) + g(x) - g(y) = f(x) + f(y) + (x-y)f\left(\frac{x+y}{2}\right) \tag{3}$$

for all $x, y \in R$ *yields a solution of the system*

$$\begin{cases} f(x+y) = f(x) + f(y) \\ g(x) - g(y) = (x-y)f\left(\frac{x+y}{2}\right) \end{cases} \tag{4}$$

for $x, y \in R$.

2. Preliminary results

Now, we study a generalization of the system (4). In order to do that we introduce a new function h in the equation (3). We prove the following

Theorem 4. *Let* $(R, +, \cdot)$ *be a uniquely 2-divisible ring. If functions* $f, g, h : R \to R$ *satisfy the functional equation*

$$f(x+y) + g(x) - h(y) = f(x) + f(y) + (x-y)f\left(\frac{x+y}{2}\right), \tag{5}$$

for all $x, y \in R$, *then there exists an additive function* $a : R \to R$ *such that*

$$\begin{cases} f(x) = a(x) + f(0) \\ g(x) = g(0) + \frac{1}{2}xa(x) + xf(0) \\ h(x) = g(0) + \frac{1}{2}xa(x) + xf(0) - f(0) \end{cases}$$

for all $x \in R$.

Proof. Replacing y by x in (5) we obtain

$$h(x) = f(2x) - 2f(x) + g(x), \quad x \in R. \tag{6}$$

Applying (6) to (5), we get

$$f(x+y) + g(x) - g(y) - f(2y) = f(x) - f(y) + (x-y)f\left(\frac{x+y}{2}\right) \tag{7}$$

for all $x, y \in R$. Let us take $y = 0$ in (7). Then,

$$g(x) = g(0) + xf\left(\frac{x}{2}\right), \quad x \in R. \tag{8}$$

Using formula (8) in (7) we get the following relation:

$$f(x+y)+xf\left(\frac{x}{2}\right)-yf\left(\frac{y}{2}\right)-f(2y) = f(x)-f(y)+(x-y)f\left(\frac{x+y}{2}\right). \tag{9}$$

Interchanging x and y, we get also

$$f(x+y)+yf\left(\frac{y}{2}\right)-xf\left(\frac{x}{2}\right)-f(2x) = f(y)-f(x)+(y-x)f\left(\frac{x+y}{2}\right). \tag{10}$$

Adding sidewise equations (9) and (10), we infer that

$$2f(x+y) = f(2x) + f(2y), \quad x, y \in R.$$

Replacing in the above equation x and y by $x/2$ and $y/2$, respectively, we have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}), \quad x, y \in R, \tag{11}$$

that is, f satisfies Jensen functional equation (see [10]), and there exists an additive function a such that $f = a + f(0)$. This together with (8) and (6) yields immediately the assertation of the theorem. □

Note that a true statement similar to the previous can be formulated:

Theorem 5. *Let $(R, +, \cdot)$ be a uniquely 2-divisible ring. If functions $f, g, h : R \to R$ satisfy the functional equation*

$$f(x+y) + g(x) - h(y) = f(x) + f(y) + (x-y)\left(\frac{f(x)+f(y)}{2}\right) \tag{12}$$

for all $x, y \in R$, then there exist an additive function $a : R \to R$ such that

$$\begin{cases} f(x) = a(x) + f(0) \\ g(x) = g(0) + \frac{1}{2}xa(x) + xf(0) \\ h(x) = g(0) + \frac{1}{2}xa(x) + xf(0) - f(0) \end{cases},$$

where $x \in R$.

Proof. Let us take $y = x$ in (12). We get

$$h(x) = f(2x) - 2f(x) + g(x), \quad x \in R. \tag{13}$$

By (13) and (12) we obtain for all $x, y \in R$

$$f(x+y)+g(x)-f(2y)-g(y) = f(x)-f(y)+(x-y)\left(\frac{f(x)+f(y)}{2}\right). \tag{14}$$

Taking $y = 0$ in the above, we deduce

$$g(x) = g(0) + \frac{1}{2}x(f(x) + f(0)), \quad x, y \in R. \tag{15}$$

Applying (15) in (14) we get

$$f(x+y) - f(2y) + \frac{1}{2}f(0)(x-y) = f(x) - f(y) + \frac{1}{2}xf(y) - \frac{1}{2}yf(x). \tag{16}$$

Interchanging x and y, we have

$$f(x+y) - f(2x) + \frac{1}{2}f(0)(y-x) = f(y) - f(x) + \frac{1}{2}yf(x) - \frac{1}{2}xf(y). \tag{17}$$

By (16) and (17) we obtain

$$2f(x+y) = f(2x) + f(2y), \quad x, y \in R.$$

Becouse of the above equation is the Jensen functional equation this completes the proof. □

3. Alienation

Assuming that $f(0) = 0$, we get the conclusion on alienation of appropriate equations.

Corollary 1. *Let $(R, +, \cdot)$ be a uniquely 2-divisible ring and $f(0) = 0$. Functions $f, g, h : R \to R$ satisfy the functional equation (5) if and only if*

$$\begin{cases} f(x+y) = f(x) + f(y) \\ g(x) - h(y) = (x-y)f\left(\frac{x+y}{2}\right) \end{cases} \tag{18}$$

for all $x, y \in R$.

Proof. It is clear that (18) implies (5).

According to Theorem 4, $h = g$ and f is an additive function. Applying (5) we deduce that

$$g(x) - g(y) = (x-y)f\left(\frac{x+y}{2}\right), \quad x, y \in R.$$

□

Corollary 2. *Let $(R, +, \cdot)$ be a uniquely 2-divisible ring and $f(0) = 0$. Functions $f, g, h : R \to R$ satisfy for all $x, y \in R$ the functional equation (12) if and only if*

$$\begin{cases} f(x+y) = f(x) + f(y) \\ g(x) - h(y) = (x-y)\left(\frac{f(x)+f(y)}{2}\right) \end{cases}, \tag{19}$$

where $x, y \in R$.

Proof. The implication (19) $\Rightarrow$ (12) is obvious.

Now, assume that functions f, g, h satisfy the equation (12). The equality $h = g$ results from Theorem 5. Moreover, the function f is additive. On account of (12),

$$g(x) - g(y) = (x - y)\left(\frac{f(x) + f(y)}{2}\right), \quad x, y \in R.$$

□

The next theorem is the main result of this paper.

Theorem 6. *Let $(R, +, \cdot)$ be a uniquely 2-divisible ring. Functions $f, g, h : R \to R$ satisfy for all $x, y \in R$ the functional equation*

$$f\left(\frac{x+y}{2}\right) + g(x) - h(y) = \frac{f(x) + f(y)}{2} + (x - y)f\left(\frac{x+y}{2}\right) \tag{20}$$

if and only if

$$\begin{cases} f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \\ g(x) - h(y) = (x - y)f\left(\frac{x+y}{2}\right) \end{cases} \tag{21}$$

for all $x, y \in R$.

Proof. It is clear that (21) implies (20).

Let us take $y = x$ in (20). We get

$$g(x) = h(x), \quad x \in R. \tag{22}$$

By means of (22) and (20), we conclude

$$f\left(\frac{x+y}{2}\right) + g(x) - g(y) = \frac{f(x) + f(y)}{2} + (x - y)f\left(\frac{x+y}{2}\right) \tag{23}$$

for every $x, y \in R$. By interchanging x and y, we obtain

$$f\left(\frac{x+y}{2}\right) + g(y) - g(x) = \frac{f(x) + f(y)}{2} + (y - x)f\left(\frac{x+y}{2}\right). \tag{24}$$

By (23) and (24) we get

$$g(x) - g(y) = (x - y)f\left(\frac{x+y}{2}\right), \quad x, y \in R.$$

Applying the above to (23), we have

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \quad x, y \in R.$$

This finishes the proof. □

Using a similar argument we receive an analogous theorem:

Theorem 7. *Let $(R, +, \cdot)$ be a uniquely 2-divisible ring. Functions $f, g, h : R \to R$ satisfy for all $x, y \in R$ the functional equation*

$$f\left(\frac{x+y}{2}\right) + g(x) - h(y) = \frac{f(x)+f(y)}{2} + (x-y)\left(\frac{f(x)+f(y)}{2}\right)$$

if and only if

$$\begin{cases} f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \\ g(x) - h(y) = (x-y)\left(\frac{f(x)+f(y)}{2}\right) \end{cases}$$

for all $x, y \in R$.

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