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ABOUT THE WAYS OF DEFINING CONNECTED SETS IN THE TOPOLOGICAL SPACES

KATARZYNA DOMAŃSKA AND MAŁGORZATA WRÓBEL

Abstract

A topological space is called connected if it is not the union of two disjoint, nonempty and open sets in this space. The standard exercises show that here the concept of open sets can be replaced by closed sets or separated sets. In this context we will discuss the definition of connected sets in topological spaces, not being the whole space with particular regard to metric spaces, without the term of subspace topology.

1. Introduction

To explain where the problem comes from, let us recal that a topological space $(X, \mathcal{T})$ is connected if it is not the union of two disjoint, nonempty, closed sets in this space, i.e., a topological space $(X, \mathcal{T})$ is connected if the conditions

$$X = A \cup B, \quad A = clA, \quad B = clB, \quad A \neq \emptyset, \quad B \neq \emptyset, \tag{1}$$

imply $A \cap B \neq \emptyset$. So, if the sets A and B satisfy (1) and $A \cap B = \emptyset$, then both $X \backslash B = A$ and $X \backslash A = B$ are also open. Therefore, in the definition of connected spaces the assumption of closedness of sets occurring in the partition of X can be replaced by the assumption of their openness. Similarly, the standard exercises show that open sets can be replaced also by separated sets (see [1], Definition 2). On the other hand, the set that is not the whole topological space is connected in a topological space if it is connected under its subspace topology. In this connection a natural question arrises: can we define a connectedness of sets without the term of subspace topology, i.e., is it true that the set is connected if it cannot be written as the union of two disjoint, nonempty and open or separated sets in the space topology?

2. Results

Let X be a nonempty set and let $(X, \mathcal{T})$ stand for the topological space. For any $A \subset X$, the closure of A will be denoted by clA and the interior of A by $intA$.

Given $M \subset X$, we denote by $(M, \mathcal{T}_M)$ the topological subspace of $(X, \mathcal{T})$ with the relative topology induced on the set M, where $\mathcal{T}_M := \{U \cap M : U \in \mathcal{T}\}$. Let $cl_M A$ and $int_M A$ stand for the closure and the interior of $A \subset M$ in $(M, \mathcal{T}_M)$, respectively. It is well known that for any topological space $(X, \mathcal{T})$ and $M \subset X$, $M \neq \emptyset$, $A \subset M$, we have

$$cl_M A = M \cap clA. \tag{2}$$

Now we give the following definitions.

Definition 1. *(see [2], Corollary 6.1.2) A topological space $(X, \mathcal{T})$ is said to be connected if it is not the union of two disjoint, nonempty and closed subsets of X.*

Definition 2. *(see [1], p. 88) Let $(X, \mathcal{T})$ be a topological space. Two subsets A and B of X are called separated in $(X, \mathcal{T})$, if*

$$A \cap clB = \emptyset \quad \text{and} \quad B \cap clA = \emptyset.$$

Notice, that two disjoint, nonempty sets are separated if, none of them has the accumulation points of the second one. In particular, two disjoint, nonempty closed or open sets are separated. Moreover, any two separated sets automatically are disjoint.

Using the standard methods we can get the following.

Remark. *Any topological space $(X, \mathcal{T})$ is connected if, and only if, X is not the union of two separated sets in $(X, \mathcal{T})$.*

A classical definition of connected set reads as follows.

Definition 3. *(see [2], p. 408) A subset M of X is called connected in the topological space $(X, \mathcal{T})$, if a subspace $(M, \mathcal{T}_M)$ is connected.*

In the course of mathematical analysis we rather rarely use a definition of a connected set as a connected subspace (as we are doing in the above definition). Now, we give two conditions, without the notion of subspace topology, equivalent to the connectivity of sets in topological spaces (Theorem 1 and Theorem 2). We will need the following lemma.

Lemma 1. *Let $(X, \mathcal{T})$ be a topological space and let $M \subset X$. Then the sets $A, B \subset M$ are separated in the subspace $(M, \mathcal{T}_M)$ if, and only if, they are separated in the space $(X, \mathcal{T})$.*

Proof. Applying (2), for any sets $A, B \subset M$, we have $cl_M A = M \cap clA$ and $cl_M B = M \cap clB$, so

$$B \cap cl_M A = B \cap (M \cap clA) = (B \cap M) \cap clA = B \cap clA$$

and

$$A \cap cl_M B = A \cap (M \cap clB) = (A \cap M) \cap clB = A \cap clB.$$

Hence, the separatedness of the sets $A, B \subset M$ in the space $(M, \mathcal{T}_M)$ (i.e., $B \cap cl_M A = \emptyset$ and $A \cap cl_M B = \emptyset$) implies their separatedness in $(X, \mathcal{T})$ (i.e., $B \cap clA = \emptyset$ and $A \cap clB = \emptyset$) and conversely. □

Now we are in a position to give a following theorem.

Theorem 1. *In any topological space $(X, \mathcal{T})$ the set $M \subset X$ is connected if, and only if, it is not the union of two separated sets in $(X, \mathcal{T})$.*

Proof. Suppose first that there exist two sets $A, B \subset X$ separated in the space $(X, \mathcal{T})$ such that $M = A \cup B$. Obviously, $A, B \subset M$ and, by Lemma 1, the sets A and B are also separated in the subspace $(M, \mathcal{T}_M)$. Thus, taking into account Remark 1, the subspace $(M, \mathcal{T}_M)$ is not connected.

Suppose now that the set $M \subset X$ is not connected in the space $(X, \mathcal{T})$, whence $(M, \mathcal{T}_M)$ is not connected space. Applying Remark 1, there exist two sets $A, B \subset M$ separated in the space $(M, \mathcal{T}_M)$ such that $M = A \cup B$. Therefore, by Lemma 1, the sets A and B are also separated in the space $(X, \mathcal{T})$, and, consequently, the set M can be written as the union of two separated sets in $(X, \mathcal{T})$, which completes the proof. □

Definition 4. *We say that a set $M \subset X$ is separated by two sets $A, B \subset X$ in the topological space $(X, \mathcal{T})$ if*

$$M \subset A \cup B, \quad A \cap B = \emptyset, \quad M \cap A \neq \emptyset, \quad M \cap B \neq \emptyset.$$

Corollary 1. *The set $M \subset X$ is connected in the topological space $(X, \mathcal{T})$ if, and only if, it can not be separated by two separated sets in this space.*

Indeed, suppose first that there exist two sets $A, B \subset X$ separated in the space $(X, \mathcal{T})$ such that $M \subset A \cup B$ and $M \cap A \neq \emptyset$, $M \cap B \neq \emptyset$. Then $M = (M \cap A) \cup (M \cap B)$ and the sets $M \cap A$ and $M \cap B$ are separated in the space $(X, \mathcal{T})$, because $cl(M \cap A) \subset clA$ and $M \cap B \subset B$, so $cl(M \cap A) \cap (M \cap B) \subset clA \cap B = \emptyset$. Thus, and by fact that $cl(M \cap B) \subset clB$ and $M \cap A \subset A$, we get $cl(M \cap B) \cap (M \cap A) \subset clB \cap A = \emptyset$. It follows that the set M can be represented as the union of two separated sets in $(X, \mathcal{T})$, which contradicts our assumption that M is connected (see Theorem 1). The converse implication is obvious.

Using the definition of connected space and by Remark 1, the space $(X, \mathcal{T})$ is connected if the set X can not be expressed as a union of two disjoint and nonempty open or closed sets, or as the union of two separated sets. In view of Corollary 1 a natural question arises: can we replace, in this corollary, separated sets by disjoint and nonempty open sets or by disjoint and nonempty closed sets? Now, we present two examples of giving a negative answer to this question.

Example 1. *Let $M = (a, b) \cup (b, c)$, where $a, b, c \in \mathbb{R}$, $a < b < c$, and let $\mathcal{T}_{d_e}$ denote a natural topology on the straight line. Although the set M can not be separated by two closed sets in the space $(\mathbb{R}, \mathcal{T}_{d_e})$, it is not connected as the union of two separated sets in $(\mathbb{R}, \mathcal{T}_{d_e})$.*

Example 2. *Let $\mathcal{T}_s$ be a topology of at most finite complements in the set of natural numbers, i.e.,*

$$\mathcal{T}_s = \left\{ U \subset \mathbb{N}\text{: } \; U = \emptyset \text{ or } \bigvee_{F \subset \mathbb{N}} (cardF < \chi_0 \;\wedge\; U = \mathbb{N} \backslash F) \right\}.$$

Then the set $M = \{1, 2\} = \{1\} \cup \{2\}$ can not be separated by two open sets in the space $(\mathbb{N}, \mathcal{T}_s)$, but in spite of this fact it is not connected, because the sets $\{1\}$ and $\{2\}$, as disjoint and closed, are separated in $(\mathbb{N}, \mathcal{T}_s)$.

However, the following two lemmas are satisfied.

Lemma 2. *If the set $M \subset X$ is connected in the topological space $(X, \mathcal{T})$ then it is not included in the union of two disjoint, nonempty and closed subsets in this space, i.e.,*

$$\sim \left(\bigvee_{A,B \subset X} \Big(A \neq \emptyset \wedge B \neq \emptyset \wedge X \backslash A \in \mathcal{T} \wedge X \backslash B \in \mathcal{T} \wedge A \cap B = \emptyset \wedge \wedge M \subset A \cup B \Big) \right).$$

Proof. Suppose that there exists two sets $A, B \subset X$ disjoint, nonempty and closed in topological space $(X, \mathcal{T})$ such that $M \subset A \cup B$. Then, $A \cap M$ and $B \cap M$ are closed in subspace $(M, \mathcal{T}_M)$, obviously they are disjoint and nonempty and $M = (A \cap M) \cup (B \cap M)$. It follows that the topological subspace $(M, \mathcal{T}_M)$ is not connected space and, by Definition 3, the set M is not connected in topological space $(X, \mathcal{T})$. □

Lemma 3. *If the set $M \subset X$ is connected in topological space $(X, \mathcal{T})$ then it is not included in the union of two disjoint, nonempty and open subsets in this space, i.e.,*

$$\sim \left(\bigvee_{A,B \subset X} \Big(A \neq \emptyset \wedge B \neq \emptyset \wedge A \in \mathcal{T} \wedge B \in \mathcal{T} \wedge A \cap B = \emptyset \wedge M \subset A \cup B \Big) \right).$$

The proof of this lemma is analogous to the proof of Lemma 2 with this difference that we use the fact that if $A, B \subset X$ are open in $(X, \mathcal{T})$, then the sets $A \cap M$ and $B \cap M$ are open in $(M, \mathcal{T}_M)$, too.

Examples 1 and 2 show that the inverses of lemmas 2 and 3 generally do not occur. However, there are some spaces in which the inverse implication occurring in Lemma 3 is true.

Let us recall the following three definitions.

Definition 5. *A topological space $(X, \mathcal{T})$ is said to be T_1 – space, if every singleton is closed, i.e.,*

$$\bigwedge_{x \in X} \Big(\{x\} = cl\{x\} \Big). \tag{3}$$

Definition 6. *(see [1], p. 56) A topological space $(X, \mathcal{T})$ is said to be normal or T_4 – space, if it is T_1 – space and*

$$\bigwedge_{\substack{E,F \subset X E = clE, F = clF \\ E \cap F = \emptyset}} \quad \bigvee_{U,V \in \mathcal{T}} (E \subset U \wedge F \subset V \wedge U \cap V = \emptyset). \tag{4}$$

Definition 7. *(see [1], p. 87) A topological space is said to be hereditary normal if every its subspace is normal.*

Before formulating the next theorem we will need the following lemma.

Lemma 4. *(see [1], Theorem 2.1.7) For any T_1-topological space $(X, \mathcal{T})$ the following three statements are equivalent:*

1. *The space $(X, \mathcal{T})$ is hereditary normal,*
2. *Every subspace of $(M, \mathcal{T}_M)$,where $\emptyset \neq M \in \mathcal{T}$, is normal,*
3. *for every pair A, B of separated subsets of X there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Theorem 2. *Let $(X, \mathcal{T})$ be a hereditary normal topological space. Then the set $M \subset X$ is connected in the space $(X, \mathcal{T})$ if, and only if, it can not be separated by two open subsets in this space.*

Proof. Suppose first that $M \subset X$ is not connected in the topological space $(X, \mathcal{T})$. Taking into account Theorem 1, we get the existence of two sets $A, B \subset X$ separated in the space $(X, \mathcal{T})$ such that $M = A \cup B$. According to Lemma 4, there exists open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$. Therefore $M \subset U \cup V$, where U, V are disjoint, nonempty and open subsets in the space $(X, \mathcal{T})$.

The converse implication follows from Lemma 3. □

Since the topological spaces with topology induced by metric are hereditary normal, we get the following corollary.

Corollary 2. *The set $M \subset X$ is connected in topological space $(X, \mathcal{T}_d)$, where $\mathcal{T}_d$ is a topology induced by the metric d, if and only if, it can not be separated by two open subsets in the general topology of this space.*

3. Conclusion

The above corollary gives a condition equivalent to the connectivity of sets in topological space with topology induced by a metric. This condition same as the condition occuring in Theorem 1 is often presented in a mathematical analysis (in which we investigate mainly metric spaces) as a definition of a connected set.

References

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Katarzyna Domańska
Jan Długosz University,
Institute of Mathematics and Computer Science,
al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
E-mail address: k.domanska@ajd.czest.pl

Małgorzata Wróbel
Częstochowa University of Technology,
Institute of Mathematics,
al. Armii Krajowej 21, 42-200 Częstochowa, Poland
E-mail address: malgorzata.wrobel@im.pcz.pl