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GENERAL APPROACH TO DETERMINING THE BASIC CHARACTERISTICS OF QUEUEING SYSTEMS WITH FINITE TOTAL CAPACITY

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Abstract. We discuss a general view of solutions for characteristics of non-classical queueing systems with random capacity customers (demands), i.e. we suppose that each customer is characterized by some random capacity (volume) and the whole capacity (total volume) of customers present in the queueing system is bounded by a constant walue $V > 0$. We determine the general view of the stationary number distribution and loss probability in the systems under consideration as compared with corresponding classical queueing systems. It's turned that in some cases we can write expressions for non-classical characteristics of finite total capacity queues if corresponding classical characteristics are known.

1. Introduction

In present work we investigate non-classical queueing systems with random volume demands and finite total (demands) capacity. It means that 1) each demand is characterized by some non-negative random space requirement (capacity or volume) ζ; 2) the total sum $\sigma(t)$ of space requirements (volumes) of all demands present in the system at arbitrary time moment t is limited by some constant value V, which is named the memory volume of the system; 3) we also assume that service time ξ of the demand and it's volume ζ are generally dependent.

Such systems have been used to model and solve various practical problems occurring in the design of computer and communicating systems.

Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the distribution function of the random vector (ζ, ξ). Then $L(x) = \mathbf{P}\{\zeta < x\} = F(x,\infty)$, $B(t) = \mathbf{P}\{\xi < t\} = F(\infty, t)$

are the distribution functions of the demand volume and service time, respectively. The space is occupied by the demand at the epoch it arrives and is released entirely at the epoch it completes service. The process $\sigma(t)$ is called the total (demands) volume.

Total volume limitation leads to additional losses of demands. A demand having the space requirement x, which arrives at the epoch τ, when there are idle servers or waiting positions, will be admitted to the system if $\sigma(\tau - 0) + x > V$. Otherwise $(\sigma(\tau - 0) + x \leq V)$, the demand will be lost.

Various queueing systems with limited memory volume were analyzed in the papers [1–8]. It follows from the papers that it's possible to determine a stationary demands number distribution and loss probability for the following queueing systems:

1) $M/M/n/(m, V)$ ($M/M/n/m$-type system in which a demand has an arbitrary distributed volume, but service time is independent of the demand volume and the total volume is limited by the value $V > 0$, $1 \leq n \leq \infty$, $0 \leq m \leq \infty$);

2) $M/G/n/(0, V)$ system (generalized Erlang system or $M/G/n/0$-type system with an arbitrary joint distribution of service time and demand volume and limited total volume);

3) processor-sharing system with an arbitrary joint distribution of service time and demand volume and limited total volume.

Our aim is to show that it is possible to determine some characteristics of non-classical (in the above sense) queueing systems, if the similar characteristics of classical systems are determined. In other words, we want to show the relation between similar classical and non-classical characteristics.

We'll demonstrate this approach by some examples.

2. M/M/n/m and M/M/n/(m, V) systems

Let a be the intensity of input flow, μ be the parameter of service time. Denote as $p_k = \mathbf{P}\{\eta = k\}$ the stationary probability of presence of k demands in the classical system, $k = \overline{0, n+m}$. Then we have the following well known equations for p_k:

$$0 = -ap_0 + \mu p_1; \tag{1}$$

$$0 = ap_0 - (a + \mu)p_1 + 2\mu p_2; \tag{2}$$

$$0 = ap_{k-1} - (a + k\mu)p_k + (k + 1)\mu p_{k+1},\ k = \overline{1, n-1}; \tag{3}$$

$$0 = ap_{k-1} - (a + n\mu)p_k + n\mu p_{k+1},\ k = \overline{1, n+m-1}; \tag{4}$$

$$0 = ap_{n+m-1} - n\mu p_{n+m}. \tag{5}$$

Suppose now that each demand in the non-classical system is characterized by some random volume ζ, and service time ξ doesn't depend on its volume. Let $L(x)$ be the distribution function of the random variable ζ. Suppose that the total demands volume is limited by value $V > 0$.

We suggest a hypothesis that probabilities $r_k = \mathbf{P}\{\eta = k\}$, $k = 0, 1, \dots,$ for the second system have the following form:

$$r_k = C p_k L_*^{(k)}(V),\ k = 1, 2, \dots,$$

where $L_*^{(k)}(x)$ is the kth order Stieltjes convolution of the function $L(x)$, i.e.

$$L_*^{(0)}(x) \equiv 1,\ L_*^{(k)}(x) = \int_0^x L_*^{(k-1)}(x-u)\,dL(u),\ k = 1, 2, \dots.$$

To conform this hypothesis, we have introduce the following functions having (for $k \geq 1$) the following probability sense:

$$g_k(x) = \mathbf{P}\{\eta = k, \sigma < x\}, \tag{6}$$

where σ is the stationary total volume of the demands present in the system. It's clear that $r_k = g_k(V)$, $k = 1, 2, \dots.$ According to our hypothesis we assume that

$$g_k(x) = C p_k L_*^{(k)}(x),\ k = 1, 2, \dots. \tag{7}$$

We obtain the following equations for introduced functions:

$$0 = -a r_0 L(V) + \mu r_1; \tag{8}$$

$$0 = a r_0 L(V) - a \int_0^V g_1(V-x)\,dL(x) - \mu r_1 + 2\mu r_2; \tag{9}$$

$$0 = a \int_0^V g_{k-1}(V-x)\,dL(x) - a \int_0^V g_k(V-x)\,dL(x) - k\mu r_k + (k+1)\mu r_{k+1},$$

$$k = \overline{1, n-1}; \tag{10}$$

$$0 = a \int_0^V g_{k-1}(V-x)\,dL(x) - a \int_0^V g_k(V-x)\,dL(x) - n\mu r_k + n\mu r_{k+1},$$

$$k = \overline{n, n+m-1}; \tag{11}$$

$$0 = a \int_0^V g_{n+m-1}(V-x)\,dL(x) - n\mu r_{n+m}. \tag{12}$$

It's easy to see that, if we substitute functions (7) to the equations (8)–(12), we obtain the equations (1)–(5) for p_k. So, our hypothesis is truthful, and we have

$$r_k = \begin{cases} \dfrac{(n\rho)^k}{k!} r_0 L_*^{(k)}(V), \ k = \overline{1, n}, \\ \dfrac{n^n \rho^k}{n!} r_0 L_*^{(k)}(V), \ \ k = \overline{n+1, n+m}. \end{cases}$$

From the normalization condition we obtain [9]

$$r_0 = \left[\sum_{k=0}^{n} \frac{(n\rho)^k}{k!} L_*^{(k)}(V) + \frac{n^n}{n!} \sum_{k=n+1}^{n+m} \rho^k L_*^{(k)}(V) \right]^{-1}.$$

For the loss probability we have [9]

$$p_l = 1 - (n\rho)^{-1} \sum_{k=1}^{n-1} k r_k - \rho^{-1} \left(1 - \sum_{k=0}^{n-1} r_k \right).$$

3. M/M/1/(∞, V) system with preemptive discipline

Let us consider $M/M/1/(\infty, V)$ system with two Poisson input flows: the first is the flow of the first priority with parameter a_1 and the second is the flow of the second priority with parameter a_2. Demands from the first flow gain an advantage over demands from the second one in accordance with preemptive resume discipline. Each demand is characterized by some random volume ζ with the distribution function $L(x)$ for the both priorities. The total demands volume $\sigma(t)$ is limited by the value $V > 0$. A demand arriving to the system will be lost in accordance with the above agreement. Demands service time doesn't depend on its volume. Let μ_1 and μ_2 be the parameter of service time of the first and second priority demands, respectively.

Our aim is to determine the stationary joint distribution of numbers of both priority demands present in the system and stationary loss probability for demands of each priority.

Let $\eta_1(t)$ and $\eta_2(t)$ be the number of demands of the first and second priority accordingly present in the system at time moment t, $\zeta_i^j(t)$ be the volume of jth demand of ith priority $(i = 1, 2)$. Then system behavior can be described by the following Markov process:

$$\left(\eta_1(t); \eta_2(t); \zeta_1^i(t), i = \overline{1, \eta_1(t)}; \zeta_2^j(t), j = \overline{1, \eta_2(t)} \right). \tag{13}$$

It's obvious that $\sigma(t) = \sum\limits_{i=1}^{\eta_1(t)} \zeta_1^i(t) + \sum\limits_{j=1}^{\eta_2(t)} \zeta_2^j(t)$.

We shall characterize the process (13) by the following functions:

$$P(0,0,t) = \mathbf{P}\{\eta_1(t) = \eta_2(t) = 0\};$$

$$G(i,j,x,t) = \mathbf{P}\{\eta_1(t) = i, \eta_2(t) = j, \sigma(t) < x\},\ i,j = 0,1,...,\ \max(i,j) \geq 1;$$

$$P(i,j,t) = \mathbf{P}\{\eta_1(t) = i, \eta_2(t) = j\} = G(i,j,V,t),\ i,j = 0,1,...,\ \max(i,j) \geq 1.$$

If stationary condition takes place ($V < \infty$), we have $\sigma(t) \Rightarrow \sigma$, $\eta_i(t) \Rightarrow \eta_i$, $i = 1,2$, in the sense of a weak convergence. Then the following limits exist:

$$p(0,0) = \lim_{t\to\infty} P(0,0,t) = \mathbf{P}\{\eta_1 = \eta_2 = 0\}; \tag{14}$$

$$g(i,j,x) = \lim_{t\to\infty} G(i,j,x,t) = \mathbf{P}\{\eta_1 = i, \eta_2 = j, \sigma < x\},$$
$$i,j = 0,1,...,\ \max(i,j) \geq 1; \tag{15}$$

$$p(i,j) = \lim_{t\to\infty} P(i,j,t) = \mathbf{P}\{\eta_1 = i, \eta_2 = j\} = g(i,j,V),$$
$$i,j = 0,1,...,\ \max(i,j) \geq 1. \tag{16}$$

It can be easy shown that the functions (14)–(16) satisfy the following equations:

$$0 = -(a_1 + a_2)p(0,0)L(V) + \mu_1 p(1,0) + \mu_2 p(0,1); \tag{17}$$

$$0 = a_1 p(0,0)L(V) - (a_1 + a_2)\int_0^V g(1,0,V-x)dL(x) - \mu_1 p(1,0) + \mu_1 p(2,0); \tag{18}$$

$$0 = a_1 \int_0^V g(i-1,0,V-x)dL(x) - (a_1 + a_2)\int_0^V g(i,0,V-x)dL(x) -$$
$$-\mu_1 p(i,0) + \mu_1 p(i+1,0),\ i = 2,3,\ldots; \tag{19}$$

$$0 = a_2 p(0,0)L(V) - (a_1 + a_2)\int_0^V g(0,1,V-x)dL(x) -$$
$$-\mu_2 p(0,1) + \mu_1 p(1,1) + \mu_2 p(0,2); \tag{20}$$

$$0 = a_2 \int_0^V g(0,j-1,V-x)dL(x) - (a_1 + a_2)\int_0^V g(0,j,V-x)dL(x) -$$
$$-\mu_2 p(0,j) + \mu_1 p(1,j) + \mu_2 p(0,j+1),\ j = 2,3,\ldots; \tag{21}$$

$$0 = a_1 \int_0^V g(i-1,j,V-x)dL(x) + a_2 \int_0^V g(i,j-1,V-x)dL(x) -$$
$$-(a_1 + a_2)\int_0^V g(i,j,V-x)dL(x) - \mu_1 p(i,j) + \mu_1 p(i+1,j),\ i,j = 1,2,\ldots, \tag{22}$$

and the following equilibrium equations take place:

$$a_1 p(0,0)L(V) = \mu_1 p(1,0),\ a_2 p(0,0)L(V) = \mu_2 p(0,1), \tag{23}$$

$$(a_1 + a_2)\int_0^V g(i,j,V-x)dL(x) = \mu_1 p(i+1,j),\ i = 1,2,\ldots,\ j = 0,1,\ldots, \tag{24}$$

$$a_2\int_0^V g(0,j,V-x)dL(x) = \mu_2 p(0,j+1),\ j = 1,2,\ldots. \tag{25}$$

For similar classical preemptive discipline system $M/M/1/\infty$ with two priority classes we can obtain for stationary functions $r(i,j) = \mathbf{P}\{\eta_1 = i, \eta_2 = j\}$, $i,j = 0,1,\ldots,$ the following known [10] equations:

$$0 = -(a_1 + a_2)r(0,0) + \mu_1 r(1,0) + \mu_2 r(0,1); \tag{26}$$

$$0 = a_1 p(0,0) - (a_1 + a_2 + \mu_1)r(1,0) + \mu_1 r(2,0); \tag{27}$$

$$0 = a_1 r(i-1,0) - (a_1 + a_2 + \mu_1)r(i,0) + \mu_1 r(i+1,0),\ i = 2,3,\ldots; \tag{28}$$

$$0 = a_2 r(0,0)L(V) - (a_1 + a_2 + \mu_2)r(0,1) + \mu_1 r(1,1) + \mu_2 r(0,2); \tag{29}$$

$$0 = a_2 r(0,j-1) - (a_1 + a_2 + \mu_2)r(0,j) + \mu_1 r(1,j) + \mu_2 r(0,j+1),\ j = 2,3,\ldots; \tag{30}$$

$$0 = a_1 r(i-1,j) + a_2 r(i,j-1) - (a_1 + a_2 + \mu_1)r(i,j) + \mu_1 r(i+1,j),\ i,j = 1,2,\ldots, \tag{31}$$

and the the following equilibrium conditions take place:

$$a_1 r(0,0) = \mu_1 r(1,0); \tag{32}$$

$$(a_1 + a_2)r(i,j) = \mu_1 r(i+1,j),\ i = 1,2,\ldots,\ j = 0,1,\ldots; \tag{33}$$

$$a_2 r(0,j) = \mu_2 p(0,j+1),\ j = 0,1,\ldots. \tag{34}$$

Assume that numbers $r(i,j)$ sutisfy equations (26)–(34) and normalization condition $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} r(i,j) = 1$. We suggest a hypothesis that

$$g(i,j,x) = Cr(i,j)L_*^{i+j}(x),\ i,j = 0,1,...,\ \max(i,j) \geq 1, \tag{35}$$

where C is some constant value. This hypothesis is truthful, as it follows from the direct substitution of the function (35) to equations (26)–(34). By this way we obtain equations (17)–(25) for the functions $r(i,j)$. So, for probabilities $p(i,j)$ we have

$$p(i,j) = Cr(i,j)L_*^{i+j}(V),\ i,j = 0,1,...,$$

where C can be obtained from the normalization condition $\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} p(i,j) = 1$.

It's known [10] that equations (26)–(34) can be solved by using the generation function $R(z_1, z_2) = \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} p(i,j) z_1^i z_2^j$. Then we have [10]

$$R(z_1, z_2) = \frac{(1-\rho)(1-\rho_1 z^*)}{(1-\rho_1 z^* - \rho_2 z_2)(1-\rho_1 z^* z_1)},$$

where $\rho_1 = a_1/\mu_1$, $\rho_2 = a_2/\mu_2$, $\rho = \rho_1 + \rho_2$,

$$z^* = \frac{a_1 + a_2(1-z_2) + \mu_1 - \sqrt{[a_1 + a_2(1-z_2) + \mu_1]^2 - 4a_1\mu_1}}{2a_1}.$$

Now we can calculate the numbers $r(i,j)$:

$$r(0,0) = 1-\rho,$$
$$r(i,j) = \frac{1}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1,z_2)\Bigg|_{z_1=z_2=0}, \quad i,j = 0,1,\ldots,\ \max(i,j) \geq 1.$$

So, the probabilities $p(i,j)$ can be determined as

$$p(0,0) = C(1-\rho),$$
$$p(i,j) = \frac{C}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1,z_2)\Bigg|_{z_1=z_2=0} L_*^{(i+j)} L(V), \quad i,j = 0,1,\ldots,\ \max(i,j) \geq 1,$$

where the constant value C can be calculated from the normalization condition, i.e.

$$C = \left[1 - \rho + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i!j!} \cdot \frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j} R(z_1,z_2)\Bigg|_{z_1=z_2=0} L_*^{(i+j)} L(V)\right]^{-1}.$$

Now we can determine loss probabilities p_l^1 and p_l^2 for both priority demands accordingly from the followin equilibrium equations:

$$a_1(1-p_l^1) = \mu_1 \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} p(i,j),\ a_2(1-p_l^2) = \mu_2 \sum_{j=1}^{\infty} p(0,j),$$

whence we have

$$p_l^1 = 1 - \frac{\mu_1}{a_1}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} p(i,j),\ p_l^2 = 1 - \frac{\mu_2}{a_2}\sum_{j=1}^{\infty} p(0,j).$$

Note that for more general case when demands from different priority have generally different volume distribution (with distribution functions $L_1(x)$ and $L_2(x)$ accordingly), the hypothesis

$$g(i,j,x) = Cr(i,j)L_{1*}^{(i)} * L_{2*}^{(j)}(x)$$

is not truth.

References

[1] E. L. Romm, V. V. Skitovich, On a generalization of Erlang problem. Avtomatika i Telemekhanika, No 6, pp. 164–167, 1971. (In Russian).

[2] A. M. Alexandrov, B. A. Kaz, Non-homogeneous demands flow service. Izvestiya AN SSSR. Tekhnicheskaya Kibernetika, No 2, pp. 47–53, 1973. (In Russian).

[3] O. M. Tikhonenko, Queueing systems with random length demands and limitations. Avtomatika i Telemekhanika, No 10, pp.126–134, 1991. (In Russian).

[4] O. M. Tikhonenko, Determination of characteristics of queueing systems with limited memory. Avtomatika i Telemekhanika, No 6, pp. 105–110, 1997. (In Russian).

[5] O. M. Tikhonenko, K. G. Klimovich, Analysis of some queueing systems with restricted summarized volume. Proc. Int. Conf. "Modern Mathematical Methods of Investigating of the Information Networks", pp. 182–186, Minsk, 2001.

[6] O. M. Tikhonenko, K. G. Klimovich, Queueing systems for random-length arrivals with limited cumulative volume. Problems of Information Transmission, vol. 37, No 1, pp. 70–79, 2001.

[7] O. M. Tikhonenko, Generalized Erlang problem in the theory of systems with random volume demands. Proc. Int. Conf. "Modern Mathematical Methods of Analysis and Optimization of Telecommunication Networks", pp. 238–243, Minsk, 2003.

[8] O. M. Tikhonenko, Generalized Erlang problem for queueing systems with limited total volume. Problems of Information Transmission, vol. 41, No. 3, pp. 243–253, 2005.

[9] O. Tikhonenko, Metody probabilistyczne analizy systemów informacyjnych. Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006.

[10] L. A. Ovcharov, Applied Queueing Theory. Mashinostroenie, Moscow, 1969. (In Russian).