Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XX (2015)
9–13
DOI http://dx.doi.org/10.16926/m.2015.20.01

VECTORS IN SCHOOL MATHEMATICS

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Abstract

Vectors have several meanings in the science. Mathematicians, physicists and other scientists make use of the notion of a vector. The notions they make use of are different in different subjects. Here we present different meanings of vectors and give some hints how to understand them in different subjects of science.

1. Introduction

Vectors at school practice have several meanings. Mathematicians, physicists and other scientists make use of the notion of a vector. The notions they make use of are different in different subjects. We shall present different meanings of vectors and give some hints how to understand them in different subjects of science. Since early 60th of previous century mathematical programs have been based on French ideas on teaching mathematics, precision up to impossibility and trying to consider all mathematics as implication of set theory.

2. Vectors at school

2.1. **Vectors in physics.** Vectors are considered by many fields of science. Before mathematicians discussed vectors on the plane (space) physicists had made use of some objects like vectors (last part of education, see [1]). They needed some objects which could present some physical moves or forces. Such objects should have something like the size, the direction and which can be represented by an arrow (see [2], [3], [4] and [5]).

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The first problem that has arisen from that kind of meaning is connected with compositions of moves which are represented by sums of vectors. But how to add two arrows? Of course these two arrows should have the same point of their origins. In such case, as the sum of two vectors means the diagonal of a parallelogram spanned by those two arrows.

When we want to enlarge the force that affects to a physical point we should have to enlarge or diminish the size of a vector, then we can enlarge or diminish the length of this vector with no change of the line which is determined by its endpoints. In this kind when we want to change the direction of the vector we have to set endpoints in their inverse direction.

This representation is quite good for description of physical moves, but not good enough for mathematical theories. Mathematical ideas arise according to the need of some applications. But then they should be defined with absolutely great precision.

2.2. **Ordered pairs of points in a space.** In the beginning of sixties of the XX century the mode which came to Poland from France was beginning the mathematics study from the most primitive notions. Set theory was a foundation of it. In view of such starting point to whole mathematics every mathematical notion had to have set theory in itself. That is why in school practice one could find a vector as an ordered pair of points of the plane or 3-dimensional space, see [4].

First let us remind the idea of ordered pairs on a plane or in 3 dimensional space. To simplify the discussion let us assume that this space is endowed with a coordinate system. Then each point from that space has 2 or 3 coordinates. It seems that each point P in the space can be identified with the sequence of 2 or 3 of its coordinates.

By a vector in a space (we assume farther that we discuss vectors in 3-dimensional space as it is the most natural) we understand an ordered pair of points from that space. Suppose that $\vec{AB}$ denotes the vector with first point A and the end point B.

Now we can define the length of a vector $\vec{AB}$: It is the length of the segment $\overline{AB}$. Hence the length of any vector $\vec{AB}$ for which $A \neq B$ is a positive number.

The vector $\vec{BA}$ is called the *opposite vector* to the vector $\vec{AB}$. Of course both of the vectors $\vec{AB}$ and BA have the same length.

For each point A the vector $\vec{AA}$ has the length equalled to zero. The vector $\vec{PP}$ for each point P in the space has coordinates equalled to zero and is called *zero vector*.

And now the problem arises: how to operate on such objects like vectors understood as ordered pairs. There is no idea how to add two ordered pairs, which operation have fundamental properties for introducing the idea of a vector. It is not obvious how to add two points in the space.

Therefore the idea of ordered pairs has to be changed.

2.3. **Coordinates of a vector, equivalent vectors.** If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, then the sequence $[b_1 - a_1, b_2 - a_2, b_3 - a_3]$ is called *coordinates of the vector* $\vec{AB}$.

Let us consider the relation $\sim$ among the vectors in the space. We say that vectors $\vec{AB}$ and $\vec{CD}$ are *equivalent* or *equal* (i.e. $\vec{AB} \sim \vec{CD}$) if these vectors have the same coordinates. It is not difficult to state that this relation is an equivalence relation, it means:

(1) $\vec{AB} \sim \vec{AB}$,
(2) if $\vec{AB} \sim \vec{CD}$, then $\vec{CD} \sim \vec{AB}$,
(3) if $\vec{AB} \sim \vec{CD}$ and $\vec{CD} \sim \vec{EF}$, then $\vec{AB} \sim \vec{EF}$.

This relation divides the set of all vectors into disjoint subsets. Each such set consists of all equivalent vectors and each vector belongs to one of those subsets (equivalence classes).

The set of all such classes is considered as the set of *free vectors* in spite to the above considered vectors which are sometimes called *located vectors*.

Of course, each located vector from the same equivalence class has the same length, all of them lay on the parallel straight lines and their arrows are pointed to the same direction.

Sometimes we identify each free vector with its located vector for which the beginning point lays in the origin of the coordinate system. The most frequent use of free vectors make us to identify those vectors as the sequence of its coordinates. Hence each such vector has the form:

$$\vec{a} = [a_1, a_2, a_3].$$

Taking two vectors $\vec{a}$, $\vec{b}$ and a real number γ we can define the sum of those vectors and the product of the vector by a number: if $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, then

$$\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3],$$

$$\gamma \cdot \vec{a} = [\gamma \cdot a_1, \gamma \cdot a_2, \gamma \cdot a_3].$$

The operations define a *vector space* (or *linear space*) over the field of real numbers.

In such a way we have defined quite different space of vectors. For theory of vector spaces see [2], [3] and [5].

The vectors of that space are (in fact) sequences of three numbers, so they have no geometrical meaning. Moreover, we came to the question: what the coordinate system means. It always make use of vectors, so where one have to apply the idea of a **vector**. Then what does mean a **vector**?

Therefore we have come to some abstract idea.

2.4. **Affine space.** The vector space $\mathbb{R}^3$, considered above, forms the objects which we call *vectors*. Points of a geometrical 3-dimensional space is another set of objects. What are the connections between them?

By the affine 3-dimensional space we understand the set of some points, say $\mathcal{E}$, vector space $\mathbb{R}^3$ and a function $\omega\colon \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}^3$, called an atlas (a system of maps) such that

$$\forall_{A\in\mathcal{E}} \forall_{\vec{a}\in\mathbb{R}^3} \exists_{B\in\mathcal{E}} \left(\omega(A,B) = \vec{a}\right),$$

$$\forall_{A\in\mathcal{E}} \forall_{B\in\mathcal{E}} \forall_{C\in\mathcal{E}} \left(\omega(A,B) + \omega(B,C) = \omega(A,C)\right).$$

Therefore, the 3-dimensional Euclidean space consists of some set of points $\mathcal{E}$, a system of maps which assign some vector to each pair of points from $\mathcal{E}$. This euclidean space is quite different from the space we had considered at school.

Moreover, we have another problem. For introducing vectors (free vectors) we applied coordinate system. But the approach to that problem is still not solved. Usually, we apply vectors to define a coordinate system. How to avoid this difficulty?

First, let us notice that the sequence of vectors $\vec{e_1}$, $\vec{e_2}$ and $\vec{e_3}$, where

$$\vec{e_1} = [1,0,0], \qquad \vec{e_2} = [0,1,0], \qquad \vec{e_3} = [0,0,1],$$

forms a base for the vector space $\mathbb{R}^3$. If we add to this system of vectors some point O in $\mathcal{E}$, then the quadruple $(O, \vec{e_1}, \vec{e_2}, \vec{e_3})$ is called the *coordinate system*.

3. Conclusions

In this way we have come to that way:

(1) First we have to introduce the two- and three-dimensional space $\mathbb{R}^2$ or $\mathbb{R}^3$.
(2) Next, we have to consider affine space of points with respect to the space $\mathbb{R}^2$ or $\mathbb{R}^3$.

And we have done. Everybody can observe now that French way into mathematics is a very special one and it is not the best way for school mathematics. Bourbaki's ideas of building the whole school mathematics on set theory is not good for school children at any age.

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Received: July 2015

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