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A NEW ITERATIVE METHOD FOR SOLUTION OF THE DUAL PROBLEM OF GEOMETRIC PROGRAMMING

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Abstract. In this article a new method of optimal solution of the dual problem is proposed. This method is based on Newton's attraction theorem. An estimate of iteration convergence is given. The method uses some new procedure of correction of the current iteration. It is shown that the method uses matrix operations at each step of calculations and has a quadric speed of the convergence.

Let us consider initial geometric programming problem without constraints:

$$y = f(x) = u(x)_1 + u(x)_2 + \ldots + u(x)_n \to \min, \tag{1}$$

where $x > 0$, $q_i = u(x)_i = C_i \prod_{j=1}^{m} x_j^{a_{ij}}$,
and also the dual problem

$$V(\delta) = \prod_{i=1}^{n} \left(\frac{C_i}{\delta_i}\right)^{\delta_i} \to \max, \tag{2}$$

$\delta \in \delta_+ = \{\delta > 0 : \ \delta^T A = 0^T, \ \delta_1 + \delta_2 + \ldots + \delta_n = 1\}$.
In (1) and (2) we have denoted: x_j and δ_i are correspondingly direct and dual variables, C_i and a_{ij} are given coefficients, at that $C_i > 0$,

$$A = (a_{ij}) = \begin{pmatrix} B \\ H \end{pmatrix},$$

where B is given squarte submatrix with determinant (by assumption) $|B| \neq 0$, and H is submatrix from such d strings of matrix A, which do not belong to the basis B. The difficulty level is characterized by this number d:$d = n - m$. Let $d > 1$.

In initial problem we need to find vector $x = x_* > 0$ with components x_{j*}, such that the value $y_* = f(x_*)$ of the criterion function $f(x)$ from (1) is minimal.

In dual problem it is necessary to find vector $\delta = \delta^* \in \delta_+$ with components $\delta_j = \delta_j^*$, which gives the maximal value V^* of the criterion function $V(\delta)$ from (2). Conditions (3) can be rewritten in the following way

$$\delta_{(m)}^T = \delta_{(d)}^T Q \quad \text{and} \quad \delta_{(d)}^T \mu = 1. \tag{3}$$

Here $\delta_{(m)}^T = (\delta_1, \ldots, \delta_m)$ and $\delta_{(d)}^T = (\delta_{m+1}, \ldots, \delta_{m+d})$ are strings from the so named basic and free elements of the vector δ, $Q = -HB^{-1}$, and μ is a vector with components μ_i, which are equal to sums of string elements in some matrix $S = (Q, I)$.

For the extra difficulty level, i.e. when d >1, in [1] the solution of the problem is given in the following analytical form:

$$y_* = \prod_{i=1}^{n} \left(\frac{C_i}{\delta_i^*} \right)^{\delta i^*}, x_{j*} = \prod_{\nu=1}^{m} \left(\frac{\delta_\nu^* y_*}{C_\nu} \right)^{k_{j\nu}}, \tag{4}$$

where numbers $\delta_i = \delta_i{}^*$ are obtained by optimal choice as a result of the solution of the dual problem (2).

At that we consider two problems: 1) to find the set δ_+ of admissible vectors δ in dual problem, 2) to find the maximum of the dual function.

Function $V(\delta)$, where the dependence $\delta^T = \delta_{(d)}^T S$ is used, is called transformed dual function and is denoted by $V(\delta_{(d)})$, where $\delta_{(d)} \in \delta_{(d)+} = \{\delta_{(d)} > 0\colon \delta_{(m)} = Q^T \delta_{(d)} > 0,\ \mu^T \delta_{(d)} = 1\}$

In [1] we have proved, that logarithm of the transformed dual function is strictly concave when $\delta_{(d)} \in \delta_{(d)+}$. If there exists a vector $\delta_{(d)} \in \delta_{(d)+}$, then the dual problem has a unique solution $\delta_{(d)} = \delta_{(d)}{}^* \in \delta_{(d)+}$, which may be determined from the optimality condition $S\tilde{z} = 0$.

To the solution $\delta_{(d)}^* \in \delta_{(d)+}$ corresponds the maximal value $V^* = V(\delta_{(d)}^*)$ of the dual function in the domain $\delta_{(d)+}$.Vector $\tilde{z}$ has elements $\tilde{z}_j = \ln(C_j/\lambda\delta_j)$, $\lambda = V^*$.

Thus, the problem is reduced to the solution of the equation $S\tilde{z} = 0$ with respect to vector $\delta_{(d)} = \delta_{(d)}^*$. In order to get it the method of fixed point may be used. However in this method the information on vector's $\delta_{(d)} = \delta_{(d)}^*$ existence and uniqueness is not taken into consideration.

In the present paper a new iterative, based on this information, method for solution of the dual problem (2) is proposed.

Following [2], consider iterations

$$x^k = G(x^{k-1}), \quad k = 1, 2, \dots, \tag{5}$$

where $G\colon D \subset R^n \to R^n$.

Vector x^* is called an attraction point of the iteration (8), if there exists open neighborhood $\Re$ of this point such, that $\Re \subset D$, and for any point $x^0 \in \Re$ all the iterations, determined by formula (8), belong to D and converge to x^*.

Note that in given definition there are no conditions specifying the set $\Re$. Therefore in every particular case (when we need to use the concept of attractive point) the set $\Re$ is specifying according to some features of the particular problem.

Let for initial data of the dual problem vector $\delta_{(d)+} \neq \varnothing$, and the initial admissible solution is already founded: $\delta^0_{(d)} \in \delta_{(d)+}$ and also the meaning $V_0 = V(\delta^0_{(d)})$. Then, as it was noted before, there exists unique optimal solution $\delta^*_{(d)} \in \delta_{(d)+}$ of the dual problem.

At that this is straight forward task to prove that the set $\delta_{(d)+}$ is open and hence can be considered as an open neighborhood of the optimal point $\delta^*_{(d)}$.

Let us input new variables with the help of the following parities

$$w_i = \delta_i V^*, \quad i = 1, \dots, n,$$

where vector of basic variables and vector of free variables:

$$w_{(m)} = (w_1, w_2, \dots, w_m)^T, \quad w_{(d)} = (w_{m+1}, w_{m+2}, \dots, w_{m+d})^T.$$

Since $w_{(m)} = V^*\delta_{(m)}$, $w_{(d)} = V^*\delta_{(d)}$, then the set

$$\Re_+ = \{w_{(d)} > 0 :\ w_{(m)} = Q^T w_{(d)} > 0\} = \{\delta_{(d)} > 0, \delta_{(m)} = Q^T \delta_{(d)} > 0\}$$

represents itself as a variety of all positive solutions of the equation $A^T\delta = 0$ and can be considered as an open neighborhood of the optimal point $w^*_{(d)} = V^*\delta^*_{(d)}$.

If the set $\delta_{(d)+} \neq \varnothing$, then by criterion of optimality the unique solution $\delta^*_{(d)}$ of the dual problem contains in $\delta_{(d)+}$, moreover, $\delta_{(d)+} \subset \Re_+$.

In case $\Re_+ \neq \varnothing$ consider equation $S\tilde{z} = 0$, rewriting it in the form

$$G(w_{(d)}) \equiv \lambda(L(w_{(d)})')^T = -Sz(w_{(d)}) = 0 \quad \text{when} \quad w_{(d)} \in \Re_+,$$

where number $\lambda = V^*$, and $L(w_{(d)})'$ — is the derivative of Lagrange's function, i.e. $L(w_{(d)})'$ is the string having elements

$$\frac{\partial L}{\partial \delta_{m+i}} = -s_i^T z, \quad i = 1, 2, \dots, d,$$

and s_i^T is the string with number i of matrix S, at that vector $z = z(w_{(d)})$ has components

$$z_i = \ln(\lambda\delta_j/C_j) = \ln(w_j/C_j), \quad j = 1, 2, \dots, n.$$

In optimality criterion a transformed Lagrange's function is considered. Therefore the vector $w_{(m)}$ is expressed by $w_{(d)} > 0$ by formula $w_{(m)} = Q^T w_{(d)}$ in assumption that $w_{(m)} = Q^T w_{(d)} > 0$, and hence $G : R^d_+ \to R^d$.

Here R^d_+ is the set of positive vectors from R^d.

It is evident that $\Re_+ \subset R^d_+$, and the mapping G: $R^d_+ \to R^d$ is differentiable in $w_{(d)}$ by Gateaux in open neighborhood $\Re_+ \subset R^d_+$, if $w_{(m)} = Q^T w_{(d)} > 0$.

By optimality criterion in point $w^*_{(d)} \in \Re_+$ the following parity is true

$$G(w^*_{(d)}) = Sz(w^*_{(d)}) = 0, \tag{6}$$

and hence vector function $G(w_{(d)})$ has components $g(w)_\nu = -s_\nu^T z(w)$, and the derivative $(g(w)_\nu)'_i$ of $g(w)_\nu$ in w_{m+i} is

$$(g(w)_\nu)'_i = -(s_{\nu 1}\ln(w_1/C_1) + \ldots + s_{\nu n}\ln(w_n/C_n))'_i = \\ = -s_\nu^T D_w s^i, \quad s^i = S^T e^i,$$

where (when $w_{(m)} = \mathrm{Q}^T w_{(d)} > 0$) the following parities hold

$$\frac{\partial w_j}{\partial w_{m+i}} = \frac{\partial \delta_j}{\partial \delta_{m+i}} = s_{ij}, \quad D_w = \mathrm{diag}(1/w_1, \dots, 1/w_n), \quad w_i = V^* \delta_i > 0.$$

At that derivative $G(w_{(d)})' = -SD_w S^T$, is continuous at point $w^*_{(d)} \in \Re_+$, and matrix $G'(w^*_{(d)})$ is not degenerate, because by supposition the rank of the matrix S is $r(S) = m$, and

$$(r(S) = m) \Rightarrow (r(SD_w S^T) = m) \Leftrightarrow (|SD_w S^T| \neq 0).$$

By theorem from [2] and taking into consideration the optimality criterion come to the conclusion: when $\Re_+ \neq \varnothing$ the solution $w^*_{(d)}$ of the equation (9) exists, it is unique and is the point of attractions for Newton's iterations

$$w^k_{(d)} = w^{k-1}_{(d)} - G'(w^{k-1}_{(d)})^{-1} G(w^{k-1}_{(d)}), \quad k = 1, 2, \dots,$$

where $G(w^{k-1}_{(d)}) = -Sz(w^{k-1}_{(d)})$, $G'(w^{k-1}_{(d)})^{-1} = -(SD(w^{k-1}_{(d)})S^T)^{-1}$, if for each vector $w^k_{(d)}$ holds

$$w^k_{(m)} = Q^T w^k_{(d)} > 0.$$

In other transcript Newton's iterations are

$$w_{(d)}^k = w_{(d)}^{k-1} - A_{k-1} z(w_{(d)}^{k-1}), \quad k = 1, 2, \dots , \tag{7}$$

where

$$A_{k-1} = (SD(w_{(d)}^{k-1})S^T)^{-1} S,$$

at that $D(w_{(d)}^{k-1}) = \operatorname{diag}(1/w_{1,k-1}, \dots , 1/w_{n,k-1})$, $z_{i,k-1} = \ln(w_{i,k-1}/C_i)$, $w_{n,k-1} > 0$, $i = 1, 2, \dots , n$, and it is assumed that

$$w_{(m)}^k = Q^T w_{(d)}^k > 0, \quad k = 1, 2, \dots . \tag{8}$$

By optimality criterion when $\Re_+ \neq \varnothing$ it is known the fact of existence of the solution $w_{(d)}^*$ of the equation $Sz(w_{(d)}) = 0$. Therefore the mentioned conclusion on adequacy of attractive point definition means that all the iterations $w_{(d)}^k$, defined by formula (10) when conditions (11) holds, for any initial point $w_{(d)}^0 \in \Re_+$ belong to R_+^d and converge to the solution $w_{(d)}^* \in \Re_+$.

Thus, iterations (10) allow to get values $w_{(d)}^k > 0$, after that it is necessary to verify inequality (11).

If $w_{(m)}^k = Q^T w_{(d)}^k > 0$, then vector $w_{(d)}^k > 0$ is admissible, i.e. $w_{(d)}^k \in \Re_+$. In this case come to the next iteration.

If the condition $w_{(m)}^k = Q^T w_{(d)}^k > 0$ does not hold, then the vector $w_{(d)}^k > 0$ is not admissible in the sense that $w_{(d)}^k \in R_+^d$, but $w_{(d)}^k \notin \Re_+$. Then we use correction method proposed in [4]. This method allows to find an admissible vector $\tilde{\delta}_{(d)}^k \in \delta_{(d)+}$ with the help of meaning $w_{(d)}^k \notin \Re_+$ if the column $\mu > 0$, and each sum of elements of every string in matrix Q^T is positive.

Theorem 1. Bearing in mind the set $\Re_+$, let us account that the vector μ>0, and each sum of elements of every string in matrix Q^T is positive.

Let the admissible initial value $\delta_{(d)}^0$ of the vector of free dual variables and also vector $w_{(d)}^0 = V_0 \delta_{(d)}^0 \in \Re_+$ are already founded. Here $V_0 = V(\delta_{(d)}^0)$ is the initial value of the dual function.

1. For any initial point $w_{(d)}^0 = V_0 \delta_{(d)}^0 \in \Re_+$, iterations defined by formula (10) of the type

$$w_{(d)}^k = w_{(d)}^{k-1} - A_{k-1} z(w_{(d)}^{k-1}), \quad k = 1, 2, \dots ,$$

converge to the unique value $w_{(d)}^* = V_{(d)}^* \delta_{(d)}^* \in \Re_+$ when the iteration's correction method is used at each iteration and the following conditions hold:

$w^k_{(m)} = Q^T w^k_{(d)} > 0, \quad k = 1, 2 \dots$. Here $\delta^*_{(d)}$ is the unique solution of the dual problem, which corresponds to the maximal value

$$V^* = V(\delta^*_{(d)}) = \mu^T w^*_{(d)} = w^*_1 + w^*_2 + \dots + w^*_n$$

of the dual function $V = V(\delta_{(d)})$.

In formulas for iterations the matrix

$$A_{k-1} = (SD(w^{k-1}_{(d)})S^T)^{-1}S, \quad \text{at that } S = (Q, I) \quad \text{and}$$

$$D(w^{k-1}_{(d)}) = \mathrm{diag}(1/w_{1,k-1}, 1/w_{2,k-1}, \dots, 1/w_{n,k-1}), \quad w_{i,k-1} > 0,$$

where

$$(w_{1k}, \dots, w_{mk}) = (w_{m+1,k}, \dots, w_{n,k})Q, \quad z(w^{k-1}_{(d)}) = (z_{1,k-1}, \dots, z_{n,k-1})^T,$$

$$z_{i,k-1} = \ln(w^{k-1}_i / C_i), \quad i = 1, 2, \dots, n.$$

2. Let we got (as a result of calculations): optimal vector $w^*_{(d)}$ and the maximal value V^* of the dual function $V = V(\delta_{(d)})$.

The optimal vector δ^* of dual variables may be computed by formulas

$$\delta^*_{(d)} = w^*_{(d)}/V^*, \quad \delta^*_{(m)} = Q^T \delta^*_{(d)}, \quad \delta^* = \begin{pmatrix} \delta^*_{(m)} \\ \delta^*_{(d)} \end{pmatrix}.$$

3. Convergence speed of iteration process (10) is quadric. This means that there exists constant such that the following inequality holds

$$\|w^k_{(d)} - w^*_{(d)}\| \leqslant c\|w^{k-1}_{(d)} - w^*_{(d)}\|^2$$

for all sufficiently large numbers k.

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