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ABOUT DIFFERENTIABILITY AND VBG_* CLASS

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Abstract

Let X be a finite dimensional real Banach space. We show that if the contingent of the curve $\Gamma\colon (a,b) \to X$ fulfils some conditions then each parametrization of that curve is VBG_*. Stanisław Saks proved that each VBG_* function is differentiable at a set of full Lebesgue measure. The result of this paper is a partial converse of that theorem.

1. Introduction

We will present a generalization of the concepts of functions of bounded variation in the restricted sense (VB_*) and of generalized bounded variation in the restricted sense (VBG_*) in the case of functions of a real variable that takes values in a real normed space. Let us recall first these definitions in the case of real-valued functions.

Definition 1. [2], [5] *If $F\colon [a,b] \to \mathbb{R}$ and $[\alpha,\beta] \subset [a,b]$, then the value*

$$\sup\big\{|F(x) - F(y)|\colon x \in [\alpha,\beta], y \in [\alpha,\beta]\big\}$$

is called an oscillation of the mapping F on the interval $[\alpha,\beta]$ and is denoted by $\omega\big(F,[\alpha,\beta]\big)$.

Definition 2. [2], [5] *If $F\colon [a,b] \to \mathbb{R}$ and $E \subset [a,b]$ then a mapping F is called of bounded variation in the restricted sense on the set E, or simply, is of VB_* on E, if*

$$\sup\sum_k \omega\big(F,[a_k,b_k]\big) < \infty,$$

where $([a_k,b_k])_{k\in\mathbb{N}}$ is any sequence of non-overlapping intervals such that $a_k \in E$, $b_k \in E$. The number $\sup\sum_k \omega\big(F,[a_k,b_k]\big)$ is denoted by $V_E F$.

Definition 2 can be generalized on the case of the mapping F with value in a real normed space X.

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Definition 3. *Let X be a real normed space and $\|\cdot\|$ be the norm in X. By the oscillation of a mapping $F\colon [a,b]\to X$ on $[\alpha,\beta]\subset[a,b]$ we call the value*

$$\sup\bigl\{\|F(x)-F(y)\|\colon x\in[\alpha,\beta], y\in[\alpha,\beta]\bigr\}.$$

This oscillation will be denoted by the symbol $\omega\bigl(F,[\alpha,\beta]\bigr)$.

Definition 4. *Let X be a real normed space and $E\subset[a,b]$. We say that a mapping $F\colon[a,b]\to X$ is VB_* on the set E, and denote $F\in VB_*(E)$, if*

$$\sup\sum_k \omega\bigl(F,[a_k,b_k]\bigr)<\infty,$$

where $([a_k,b_k])_{k\in\mathbb{N}}$ is any sequence of non-overlapping intervals such that $a_k\in E$, $b_k\in E$. The value $\sup\sum_k\omega\bigl(F,[a_k,b_k]\bigr)$ is denoted by $V_E F$.

Now we assume that dimension of X is finite. Observe that the fact that F is VB_* on some set is independent of the choice of a norm in X. Let $F\colon[a,b]\to X$ and $e=(e_1,\dots,e_n)$ be a base of the space X. Then

$$F=\sum_{i=1}^{n}F_i e_i.$$

Mappings F_i are called coordinates of the mapping F with respect to the base e. We also shall use denotation $F=(F_1,\dots,F_n)$. Straightforward calculations prove the next lemma.

Lemma 1. *If X is a finite dimensional real normed space, $F\colon[a,b]\to X$ and $E\subset[a,b]$, then:*

(1) *If F is VB_* on E then for each base $e=(e_1,\dots,e_n)$ of the space X mappings F_i are VB_* on E for each $i\in\{1,\dots,n\}$.*
(2) *If there exists a base $e=(e_1,\dots,e_n)$ of the space X for which mappings F_i, $i\in\{1,\dots,n\}$, are VB_* on the set E then $F\in VB_*(E)$.*

Definition 5. [2], [5] *Let $E\subset[a,b]$. We say that a mapping $F\colon[a,b]\to\mathbb{R}$ is of generalized bounded variation in the restricted sense on E, or simply, is VBG_* on the set E, and denote $F\in VBG_*(E)$, if E is a countable union of sets on each of which the mapping F is VB_*.*

We can generalize this definition in the following way:

Definition 6. *Let X be a real normed space, $E\subset[a,b]$. We will say that a mapping $F\colon[a,b]\to X$ is VBG_* on E, and denote $F\in VBG_*(E)$, if E is a countable union of sets such that for each of them F is VB_*.*

The proof of the next lemma is technical, we shall omit it.

Lemma 2. *Let X be a real normed space,* $\dim X=n$, *$F\colon[a,b]\to X$ and $E\subset[a,b]$. Then*

(1) *If* F *is* VBG_* *on* E *then for each base* $e = (e_1, \dots, e_n)$ *of the space* X *mappings* F_i *are* VBG_* *on the set* E *for each* $i \in \{1, \dots, n\}$.
(2) *If there exists a base* $e = (e_1, \dots, e_n)$ *of the space* X *for which each mapping* F_i, $i \in \{1, \dots, n\}$, *is* VBG_* *on* E *then* F *is* VBG_* *on* E.

Theorem 1. [5] *Let* $E \subset [a, b]$. *If a function* $F\colon [a, b] \to \mathbb{R}$ *is* VBG_* *on the set* E, *then* F *is differentiable at a set of full Lebesgue measure.*

The obvious corollary of this theorem for a mappings which take values in a real normed space is as follows:

Corollary 1. *Let* X *be a real normed space,* $\dim X < \infty$ *and* $E \subset [a, b]$. *If a mapping* $F\colon [a, b] \to X$ *is* VBG_* *on the set* E, *then* F *is differentiable (in the Fréchet sense) at almost all points of this set.*

Definition 7. [6] *Let* $\emptyset \neq M \subset Z$, *where* Z *is a real normed space. Let* z *belong to the closure of the set* M. *The set*

$$\Big\{v \in Z\colon \exists (z_n)_{n\in\mathbb{N}}, z_n \in M, \lim_{n\to\infty} z_n = z, \exists (\lambda_n)_{n\in\mathbb{N}}, \lambda_n > 0\colon \lim_{n\to\infty} \lambda_n (z_n - z) = v\Big\}$$

is called the tangent cone to M *at* z *and is denoted by* $\mathrm{Tan}(M, z)$. *The elements of* $\mathrm{Tan}(M, z)$ *are called vectors tangent to* M *at* z. *The set* $\mathrm{Tan}(M, z)$ *is also called the contingent of* M *at* z *(see* [1], [5]*).*

The basic properties of the contingent and the connections between differentiability of a mapping $f\colon X \to Y$ at a point, where X, Y are real normed spaces and the contingent of its graph one can find in [3], [4], [6], [7].

Definition 8. *If* X *is a real normed space, then a mapping* f *is called an embedding if it is a homeomorphism of the interval* (a, b) *into* X, *where* $f\big((a, b)\big)$ *is equipped with the subspace topology. A subset* Γ *of the space* X *is called a curve if there is an embedding* $f\colon (a, b) \to X$ *such that* $f\big((a, b)\big) = \Gamma$. *This embedding is called a parametrization of the curve* Γ.

The following theorem gives a connection between the contingent of a curve and the existence of a differentiable parametrization of this curve.

Theorem 2. [8] *Let* X *be a real normed space for which* $1 < \dim X < \infty$. *Assume that for a curve* $\Gamma \subset X$ *the following conditions are fulfilled:*
(i) *for each* $p \in \Gamma$ *the contingent* $\mathrm{Tan}(\Gamma, p)$ *is one-dimensional linear subspace of* X,
(ii) *there exists a subspace* Y *of* X *such that* $\mathrm{codim} Y = 1$ *and*

$$\mathrm{Tan}(\Gamma, p) \not\subset Y$$

for each $p \in \Gamma$.

Then there exist an open interval (c,d) and a differentiable parametrization $g\colon (c,d) \to \Gamma$ of the curve Γ such that

$$\inf_{t\in(c,d)} \left\|g'(t)\right\| > 0.$$

Corollary 2. [8] *Let $f\colon (a,b) \to \Gamma$ be a parametrization of the curve Γ. Then under assumptions of theorem 2, for every open interval (c,d) there exists a mapping $g\colon (c,d) \to \Gamma$ such that the mapping $g^{-1} \circ f\colon (a,b) \to (c,d)$ is an increasing homeomorphism.*

Corollary 3. [8] *Under assumptions of theorem 2, each parametrization of the curve Γ is almost everywhere differentiable.*

Theorem 3. [2] *A mapping $F\colon [0,1] \to \mathbb{R}$ is continuous and VBG_* on the interval $[0,1]$ if and only if there exists a homeomorphism $h\colon [0,1] \to [0,1]$ such that $F \circ h$ is differentiable.*

We will use the following easy generalization of theorem 3.

Theorem 4. *Let $F\colon [0,1] \to \mathbb{R}$. The mapping F is continuous and VBG_* on $[0,1]$ if and only if there exists a homeomorphism $h\colon [c,d] \to [0,1]$ such that $F \circ h$ is differentiable.*

2. Main results

Applying theorem 2., lemma 2. and theorem 4. we will prove that each parametrization of a curve Γ satisfying assumptions of theorem 2 is VBG_*. The following theorem is a partial converse of the corollary 1.

Theorem 5. *Let X be a real normed space such that $1 < \dim X < \infty$. Assume that for a curve $\Gamma \subset X$ the following conditions are fulfilled:*

- (i) *for each $p \in \Gamma$ the contingent $\mathrm{Tan}(\Gamma, p)$ is one-dimensional linear subspace of X,*
- (ii) *there exists a subspace Y of X such that $\mathrm{codim} Y = 1$ and*

$$\mathrm{Tan}(\Gamma, p) \not\subset Y$$

for each $p \in \Gamma$.

Then each parametrization $f\colon (a,b) \to \Gamma$ of the curve Γ is VBG_* in (a,b).

Proof. Let $f\colon (a,b) \to \Gamma$ be a parametrization of the curve Γ. By theorem 2, there exists a differentiable parametrization $g\colon (c,d) \to \Gamma$ of that curve. Obviously $f^{-1} \circ g$ is a homeomorphism of (c,d) onto (a,b).

Fix an interval $[c_1, d_1]$ contained in (c,d). Then there exists an interval $[a_1, b_1]$ in the set (a,b) such that the mapping

$$\big(f^{-1}\circ g\big)\,|_{[c_1,d_1]}\colon [c_1,d_1]\to[a_1,b_1]$$

is a homeomorphism of $[c_1,d_1]$ onto $[a_1,b_1]$.

Denote $h^*=\big(f^{-1}\circ g\big)\,|_{[c_1,d_1]}$, $f^*=f|_{[a_1,b_1]}$ and $g^*=g|_{[c_1,d_1]}$. Obviously, f^* is continuous and g^* is differentiable.

Fix a base $e=(e_1,\dots,e_n)$ of the space X. Then

$$f^*(t)=\sum_{i=1}^{n} f_i^*(t)e_i \quad\text{and}\quad g^*(\tau)=\sum_{i=1}^{n} g_i^*(\tau)e_i,$$

where $f_i^*\colon[a_1,b_1]\to\mathbb{R}$, $g_i^*\colon[c_1,d_1]\to\mathbb{R}$, $i\in\{1,\dots,n\}$ and $t\in[a_1,b_1]$, $\tau\in[c_1,d_1]$. Since $g^*=f^*\circ h^*$, then $g_i^*=f_i^*\circ h^*$ for each $i\in\{1,\dots,n\}$. The mapping g^* is differentiable, so g_i^* is differentiable if $i\in\{1,\dots,n\}$.

Moreover, h^* is a homeomorphism and f_i^* is continuous if $i\in\{1,\dots,n\}$ and by theorem 4 we have $f_i^*\in VBG_*\big([a_1,b_1]\big)$ for each $i\in\{1,\dots,n\}$.

By lemma 2(2) we conclude that the mapping

$$f^*\colon[a_1,b_1]\to X$$

is VBG_* in $[a_1,b_1]$. Therefore the mapping f is VBG_* on each closed subinterval of (a,b). The interval (a,b) is a countable union of closed subintervals, so the mapping f is VBG_* on (a,b). □

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