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K-CONTINUITY OF K-SUBQUADRATIC SET-VALUED FUNCTIONS

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Abstract

Let $X = (X, +)$ be an arbitrary topological group. A set-valued function $F\colon X \to n(Y)$ is called K-subquadratic if

$$2F(s) + 2F(t) \subset F(s+t) + F(s-t) + K,$$

for all $s,\, t \in X$, where Y denotes a topological vector space and where K is a cone in this space.

In this paper the K-continuity problem of multifunctions of this kind will be considered with respect to weakly K-boundedness. The case where $Y = \mathbb{R}^N$ will be considered separately.

1. Introduction

Let $X = (X, +)$ be an arbitrary topological group. A real-valued function F, is called subquadratic, if it fulfils inequality

$$F(x+y) + F(x-y) \leq 2F(x) + 2F(y), \qquad x, y \in X. \tag{1}$$

If the sign "$\leq$" in (1) is replaced by "$\geq$" then F is called superquadratic. The continuity problem of functions of this kind was considered in [1]. This problem can be also considered in the class of set-valued functions. Then we have two inclusions

$$F(x+y) + F(x-y) \subset 2F(x) + 2F(y), \qquad x,\ y \in X \tag{2}$$

and

$$2F(x) + 2F(y) \subset F(x+y) + F(x-y), \qquad x,\ y \in X. \tag{3}$$

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where $F\colon X \to n(Y)$ and where Y denotes a topological vector space. The continuity problem of set-valued functions defined by inclusions (2) and (3) was considered in [4] and [5].

Adding a cone K in the space of values let us consider a K-subquadratic set-valued function F, that is solution of the inclusion

$$(4) \qquad 2F(x) + 2F(y) \subset F(x+y) + F(x-y) + K, \qquad x, y \in X$$

which is defined on 2-divisible topological group X with non-empty, compact and convex values in a locally convex topological vector space Y. The K-continuity problem of multifunctions of this kind was considered in [6]. Here the K-continuity problem of K-subquadratic set-valued functions will be considered with respect to the weakly K-boundedness. In the last part of this paper we will present some conditions which imply K-continuity of K-subquadratic multifunctions which values are in $n(\mathbb{R}^N)$.

The concept of K-subquadraticity is related to real-valued subquadratic functions. In case when F is a real single-valued function and $K = [0, \infty)$, we obtain the standard definition of subquadratic functionals (1). Assuming $K = \{0\}$ in (4) we obtain the inclusion (3).

Let us start with the notations used in this paper. Let Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y, $cc(Y)$ – the family of all compact and convex members of $n(Y)$, $B(Y)$ – the family of all bounded members of $n(Y)$ and $Bcc(Y)$ – the family of all bounded, compact and convex members of $n(Y)$. The term set-valued function will be abbreviated to the form s.v.f.

First of all we shall present some definitions for the sake of completeness. Recall that a set $K \subset Y$ is called a cone if $K + K \subset K$ and $sK \subset K$ for all $s \in (0, \infty)$.

Definition 1. (cf. [2]) *A cone K in a topological vector space Y is said to be a normal cone if there exists a base $\mathfrak{W}$ of zero in Y such that*

$$W = (W + K) \cap (W - K)$$

for all $W \in \mathfrak{W}$.

Definition 2. (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-upper semicontinuous (abbreviated K-u.s.c.) at $x_0 \in X$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that*

$$F(x) \subset F(x_0) + V + K$$

for every $x \in x_0 + U$.

Definition 3. (cf. [2]) *An s.v.f. $F\colon X \to n(Y)$ is said to be K-lower semicontinuous (abbreviated K-l.s.c.) at $x_0 \in X$ if for every neighbourhood V*

of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x_0) \subset F(x) + V + K$$

for every $x \in x_0 + U$.

Definition 4. (cf. [2]) *An s.v.f. $F: X \to n(Y)$ is said to be K-continuous at $x_0 \in X$ if it is both K-u.s.c. and K-l.s.c. at x_0. It is said to be K-continuous if it is K-continuous at each point of X.*

Note that in the case where $K = \{0\}$ the K-continuity of F means its continuity with respect to the Hausdorff topology on $n(Y)$.

In our proofs we use known following lemma.

Lemma 1. (cf. [6]) *Let Y be a topological vector space and K be a cone in Y. Let A, B, C be non-empty subsets of Y such that $A + C \subset B + C + K$. If B is convex and C is bounded then $A \subset \overline{B + K}$.*

In our proofs we will also use two known lemmas (see Lemma 1.6 and Lemma 1.5 in [2]). The first lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set, where Y denotes a real topological vector space, then the set $A + B$ is closed. The second lemma says that for any bounded sets $A, B \subset Y$, where Y denotes the same space as above, the set $A + B$ is bounded.

Let us adopt the following three definitions which are natural extension of the concept of the boundedness for real-valued functions.

Definition 5. *An s.v.f. $F: X \to n(Y)$ is said to be K-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B + K$ for all $x \in A$.*

Definition 6. *An s.v.f. $F: X \to n(Y)$ is said to be K-upper bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that $F(x) \subset B - K$ for all $x \in A$.*

Definition 7. *An s.v.f. $F: X \to n(Y)$ is said to be locally K-bounded in X if for every $x \in X$ there exists a neighbourhood U_x of zero in X such that F is K-lower and K-upper bounded on a set $x + U_x$.*

2. The main result connected with weakly K-boundedness

Let us introduce the following definitions:

Definition 8. *An s.v.f. $F: X \to n(Y)$ is said to be weakly K-lower bounded on a set $A \subset X$ if there exists a bounded set $B \subset Y$ such that*

$$F(x) \cap (B + K) \neq \emptyset$$

for all $x \in A$.

Definition 9. *An s.v.f.* $F\colon X \to n(Y)$ *is said to be weakly* K*-upper bounded on a set* $A \subset X$ *if there exists a bounded set* $B \subset Y$ *such that*

$$F(x) \cap (B - K) \neq \emptyset$$

for all $x \in A$.

Definition 10. *An s.v.f.* $F\colon X \to n(Y)$ *is said to be locally weakly* K*-bounded in* X *if for every* $x \in X$ *there exists a neighbourhood* U_x *of zero in* X *such that* F *is weakly* K*-lower and weakly* K*-upper bounded on a set* $x + U_x$.

Clearly, if F is K-upper (K-lower) bounded on a set A, then it is weakly K-upper (K-lower) bounded on a set A. In the case of single-valued functions these definitions coincide.

Definition 11. *We say that* 2*-divisible topological group* X *has the property* $\left(\frac{1}{2}\right)$ *if for every neighbourhood* V *of zero there exists a neighbourhood* W *of zero such that* $\frac{1}{2}W \subset W \subset V$.

For the K-subquadratic set-valued functions the following theorem holds.

Theorem 1. (cf. [6]) *Let* X *be a* 2*-divisible topological group satisfying condition* $\left(\frac{1}{2}\right)$, Y *– locally convex topological vector space and a subset* K *of* Y *– a closed normal cone. If a* K*-subquadratic s.v.f.* $F\colon X \to cc(Y)$ *is* K*-continuous at zero, locally* K*-bounded in* X *and* $F(0) = \{0\}$, *then it is* K*-continuous in* X.

Lemma 2. *Let* X *be a* 2*-divisible topological group satisfying condition* $\left(\frac{1}{2}\right)$, Y *– topological vector space and* $K \subset Y$ *a cone. Let* $F\colon X \to B(Y)$ *be a* K*-subquadratic s.v.f. such that* $F(0) = \{0\}$ *and* $G\colon X \to n(Y)$ *be an s.v.f. with*

$$G(x) \subset F(x) + K \tag{5}$$

for all $x \in X$.

If F *is* K*-lower bounded at zero and* G *is locally weakly* K*-upper bounded in* X, *then* F *is locally* K*-lower bounded in* X.

Proof. Let $x \in X$. There exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x - t) \cap (B_1 - K) \neq \emptyset, \qquad t \in U_1,$$

which implies that

$$0 \in G(x - t) - B_1 + K \tag{6}$$

for all $t \in U_1$. Since F is K-lower bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2 + K, \qquad t \in U_2. \tag{7}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (5), (6) i (7), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1+K \subset F(x+t)+F(x-t)-B_1+K \subset$$
$$\subset \frac{1}{2}F(2x)+\frac{1}{2}F(2t)-B_1+K \subset \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1+K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B}+K, \qquad t \in \widetilde{U},$$

which means that F is locally K-lower bounded in X. □

Lemma 3. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y topological vector space and $K \subset Y$ a cone. Let $F\colon X \to B(Y)$ be a K-subquadratic s.v.f. such that $F(0)=\{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset F(x)-K \tag{8}$$

for all $x \in X$.

If F is K-upper bounded at zero and G is locally weakly K-lower bounded in X, then F is locally K-upper bounded in X.

Proof. Let $x \in X$. Since G is weakly K-lower bounded in x, then there exist a bounded set $B_1 \subset Y$ and a symmetric neighbourhood U_1 of zero in X such that

$$G(x-t) \cap (B_1+K) \neq \emptyset, \qquad t \in U_1,$$

which implies that

$$0 \in G(x-t)-B_1-K \tag{9}$$

for all $t \in U_1$. Since F is K-upper bounded at zero, there exist a symmetric neighbourhood U_2 of zero in X and a bounded set $B_2 \subset Y$ such that

$$F(t) \subset B_2-K, \qquad t \in U_2. \tag{10}$$

Let $\widetilde{U}$ be a symmetric neighbourhood of zero in X with $\frac{1}{2}\widetilde{U} \subset \widetilde{U} \subset U_1 \cap U_2$. Let $t \in \frac{1}{2}\widetilde{U}$. Using (8), (9) i (10), we obtain

$$F(x+t)+0 \subset F(x+t)+G(x-t)-B_1-K \subset F(x+t)+F(x-t)-B_1-K \subset$$
$$\subset \frac{1}{2}F(2x)+\frac{1}{2}F(2t)-B_1-K \subset \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1-K.$$

Define $\widetilde{B} := \frac{1}{2}F(2x)+\frac{1}{2}B_2-B_1$. Since $F(2x)$ is a bounded set, then the set $\widetilde{B}$ is also bounded as the sum of bounded sets. Therefore

$$F(x+t) \subset \widetilde{B}-K, \qquad t \in \widetilde{U},$$

which means that F is locally K-upper bounded in X. □

As an immediate consequence of Lemma 2 and Lemma 3 we obtain the following lemma.

Lemma 4. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y topological vector space and $K \subset Y$ a cone with zero. Let $F\colon X \to B(Y)$ be a K-subquadratic s.v.f. such that $F(0) = \{0\}$. If F is K- bounded at zero and locally weakly K-bounded in X, then it is locally K-bounded in X.*

Let us note, that Theorem 1, Lemma 2 and Lemma 3 yield directly the following result.

Theorem 2. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$ and $G\colon X \to n(Y)$ be an s.v.f. with*

$$G(x) \subset (F(x) - K) \cap (F(x) + K)$$

for all $x \in X$.

If F is K-bounded at zero and K-continuous at zero, G is locally weakly K-bounded in X, then F is K-continuous everywhere in X.

Remark 1. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. Let $F\colon X \to Bcc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$.*

If F is K-continuous at zero, K- bounded at zero and locally weakly K-bounded in X, then it is K-continuous in X.

Proof. Note that a closed cone is a cone with zero. Then the following inclusion

$$F(x) \subset (F(x) - K) \cap (F(x) + K)$$

holds for all $x \in X$. Using Theorem 2 for $G = F$ we end the proof. □

3. The case $Y = \mathbb{R}^N$

Now we consider the case where the space of values is $n(\mathbb{R}^N)$. It is known that for K-subquadratic set-valued functions the following lemma holds.

Lemma 5. (cf. [6]) *Let X be a 2-divisible topological group, Y locally convex topological vector space and $K \subset Y$ a closed normal cone. If a K-subquadratic s.v.f. $F\colon X \to cc(Y)$ is K-continuous at zero, $F(0) = \{0\}$ and locally K-lower bounded in X, then it is K-u.s.c. in X.*

In this part of the paper Y will be denote $\mathbb{R}^N$.

Theorem 3. *Let X be a 2-divisible topological group and K be a closed normal cone in Y. Let $F\colon X \to cc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$. If F is K-continuous at zero and locally K-lower bounded in X, then it is K-continuous in X.*

Proof. By Lemma 5 F is K-u.s.c. in X. Now we will show that F is K-l.s.c. in X. Let $x_0 \in X$ and let V be a neighbourhood of zero in Y. There exists a convex neighbourhood W of zero in Y such that the set $\overline{W}$ is compact with $3\overline{W} \subset V$. Since F is K-u.s.c. at x_0 then there exists a symmetric neighbourhood U of zero in X such that

$$F(x_0 + t) \subset F(x_0) + W + K, \tag{11}$$

$$F(x_0 - t) \subset F(x_0) + W + K, \tag{12}$$

for all $t \in U$.
Since F is K-l.s.c. at zero and $F(0) = \{0\}$, there exists a neighbourhood U_0 of zero in X such that

$$\{0\} \subset F(t) + W + K \qquad t \in U_0. \tag{13}$$

Consider a symmetric neighbourhood $\widetilde{U}$ of zero in X with $\widetilde{U} \subset U \cap U_0$. Let $t \in \widetilde{U}$. Using (12) i (13), we obtain

$$F(x_0)+\{0\} \subset F(x_0)+F(t)+W+K \subset \frac{1}{2}F(x_0+t)+\frac{1}{2}F(x_0-t)+W+K \subset$$
$$\subset \frac{1}{2}F(x_0+t) + \frac{1}{2}F(x_0) + \frac{3}{2}\overline{W} + K$$

By convexity of the set $F(x_0)$, we have

$$\frac{1}{2}F(x_0) + \frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0) + \frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K.$$

Note that the set $\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W}$ is convex and compact. Therefore, the set $\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K$ is closed. Using Lemma 1

$$\frac{1}{2}F(x_0) \subset \overline{\frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K} = \frac{1}{2}F(x_0+t) + \frac{3}{2}\overline{W} + K,$$

and consequently

$$F(x_0) \subset F(x_0+t) + 3\overline{W} + K \subset F(x_0+t) + V + K,$$

for all $t \in \widetilde{U}$. It means that F is K-l.s.c. in X. □

Theorem 4. *Let X be a 2-divisible topological group satisfying condition $\left(\frac{1}{2}\right)$ and K be a closed normal cone in $\mathbb{R}^N$. Let $F\colon X \to cc(Y)$ be a K-subquadratic s.v.f. with $F(0) = \{0\}$. If F is K-continuous at zero and locally weakly K-upper bounded in X, then it is K-continuous in X.*

Proof. Let V be a bounded neighbourhood of zero in Y. Since F is K-u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood U of zero in X such that

$$F(t) \subset V + K, \qquad t \in U.$$

It means that F is K-lower bounded at zero. By Lemma 2 (with $G = F$), F is locally K-lower bounded in X. Applying Theorem 3, F is K-continuous in X. $\square$

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