Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XII, Częstochowa 2007

PROPERTIES OF FOURIER COEFFICIENTS OF SPLINE WAVELETS

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Abstract. Periodic B-spline functions have got many useful properties. Especially it is the property of its Fourier coefficients. In this article it is introduced and proved a similar property of Fourier coefficients of spline wavelets.

1. Spline wavelets

In this section, we shall briefly summarize the essences of the theory of the wavelet expansion. We start by defining of a multiresolution analysis.

Definition: The multiresolution analysis of $L^2(\mathbb{R})$ is a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$, with the following properties:

[1] $V_j \subset V_{j+1}$

[2] $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

[3] $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$

[4] $\bigcup\limits_{j=-\infty}^{+\infty} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap\limits_{j=-\infty}^{+\infty} V_j = \{0\}$

[5] A scaling function $\phi \in V_0$, with a non-vanishing integral, exists such that the collection $\{\phi(x-l) | l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Since $\phi \in V_0 \subset V_1$, a sequence $(h_k) \in \ell^2(\mathbb{Z})$ exists such that the scaling function satisfies the dilation equation

$$\phi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2x-k) \tag{1}$$

It is immediately to view that the collection of functions $\{\phi_{j,k}| \quad k \in \mathbb{Z}\}$, with $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$, is a Riesz basis of V_j.

We will define a space W_j complementing V_j in V_{j+1}, i.e. a space that satisfies

$$V_{j+1} = V_j \oplus W_j,$$

where symbol $\oplus$ stands for direct sum. From this follows the relation $\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$. This subspace W_j is called "wavelet subspace" and is generated by $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, where function $\psi(x)$ is called the "wavelet" and collection of functions $\{\psi(x-k)|k \in \mathbb{Z}\}$ forms a Riesz basis of W_0.

Since the wavelet ψ is an element of V_1, a sequence $(g_k) \in \ell^2(\mathbb{Z})$ exists such that

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k \phi(2x-k). \tag{2}$$

We described wavelets and scaling functions defined on the real line. For many applications it is necessary, or at least more natural, to work on a subset of the real line. Many of these cases can be dealt with by introducing periodized scaling function and wavelets, which we define as follows:

$$\phi_j^{\langle 0,1\rangle}(x) = 2^{\frac{j}{2}}\sum_{n\in\mathbb{Z}} \phi(2^j(x-n)) \tag{3}$$

$$\psi_j^{\langle 0,1\rangle}(x) = 2^{\frac{j}{2}}\sum_{n\in\mathbb{Z}} \psi(2^j(x-n)). \tag{4}$$

$$\tag{5}$$

Functions $\phi_j^{\langle 0,1\rangle}(x-k2^{-j})$ and $\psi_j^{\langle 0,1\rangle}(x-k2^{-j})$ for $k = 0, 1, .., 2^j - 1$ are linear independent and generate the spaces $V_j^{\langle 0,1\rangle}$ eventually $W_j^{\langle 0,1\rangle}$. Let us define for $j \in \mathbb{N}, j \geq j_0$ and $n_j = 2^j$ the set $K_j = \{0, 1, .., n_j - 1\}$ and take equidistant partition of the interval $\langle 0, 1\rangle$ with step width $h_j = \frac{1}{n_j}$. The points x_k^j are defined by $x_k^j = \frac{k}{n_j}$ for $k \in K_j$.
Characteristic function on the interval $\langle 0, 1\rangle$ is defined as

$$\chi = \begin{cases} 1 & x \in \langle 0, 1\rangle \\ 0 & \text{elsewhere} \end{cases}$$

For $r > 1$ we define χ^r as the convolution

$$\chi^r(x) = \int_0^1 \chi^{r-1}(x-y)\chi(y)dy.$$

This function is B-spline on $\mathbb{R}$ of the order r. If we take $\phi(x) = \chi^r(x)$, where $\phi(x)$ is scaling function introduced above, then $\phi_j^{\langle 0,1\rangle}$ for $k \in K_j$ is B-spline of the order r (piecewise polynomial of order $r-1$) on the interval $\langle 0,1\rangle$. To this scaling function we can build (as was shown earlier) functions $\psi_j^{\langle 0,1\rangle}(x-k2^{-j})$.

2. Property of Fourier coefficient of B-spline and spline wavelets

Fourier series of one-periodic function $\phi_j^{\langle 0,1\rangle}$ have form

$$\phi_j^{\langle 0,1\rangle}(x) = \sum_{p\in\mathbb{Z}} \widehat{\phi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x},$$

where $\widehat{\phi}_j^{\langle 0,1\rangle}(p)$ are Fourier coefficient defined by formula

$$\widehat{\phi}_j^{\langle 0,1\rangle}(p) = \int_0^1 \phi_j^{\langle 0,1\rangle}(x) e^{-2\pi i p x} dx. \tag{6}$$

Theorem 1. Let S_n^d is the space of one-periodic B-spline of order $d+1$ (piecewise polynomial of degree d) with knots $x_k = \frac{k}{n}$, where n is arbitrary natural number, $j = 0, ..., n-1$. Let us define a set $\Lambda_n = \{p \in \mathbb{Z}; -\frac{n}{2} < p \leq \frac{n}{2}\}$. Then for Fourier coefficients of one-periodic B-spline $\widehat{\phi}$ equality holds

$$(-1)^{l(d+1)}\widehat{\phi}(p+ln)(p+ln)^{d+1} = \widehat{\phi}(p)p^{d+1}, \qquad l \in \mathbb{Z}, \phi \in S_n^d, p \in \Lambda_n. \tag{7}$$

This equality was proved in [1].

Relation between Fourier coefficients of functions $\phi_j^{\langle 0,1\rangle}$ and $\phi_{j+1}^{\langle 0,1\rangle}$ arise from dilation equation by the following way:

$$\begin{aligned}
\sum_{p\in\mathbb{Z}} \widehat{\phi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x} &= \phi_j^{\langle 0,1\rangle}(x) \\
&= \sum_{k\in\mathbb{Z}} h_k \phi_{j+1}^{\langle 0,1\rangle}(x - k2^{-j-1}) \\
&= \sum_{k\in\mathbb{Z}} h_k \left(\sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{2\pi i p (x-k2^{-j-1})} \right) \\
&= \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) \left(\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p k 2^{-j-1}} \right) e^{2\pi i p x}.
\end{aligned}$$

By comparing of Fourier coefficients we obtain

$$\widehat{\phi}_j^{\langle 0,1\rangle}(p) = \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) m_{j+1}(p), \tag{8}$$

where $m_{j+1}(p) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p k 2^{-j-1}}$, is function with period 2^{j+1}. It is valid for this function

$$m_{j+1}(2p) = m_j(p) \tag{9}$$

and

$$m_{j+1}(2^j) = 0. \tag{10}$$

Using this relation, we can prove following theorem.

Theorem 2. Let $\phi_j^{\langle 0,1\rangle}$ is a member of space $V_j^{\langle 0,1\rangle} \equiv S_{n_j}^d$ one-periodic B-spline of order $d+1$, where $n_j = 2^j$ and $\psi_j^{\langle 0,1\rangle}$ is corresponding spline wavelet from the space $W_j^{\langle 0,1\rangle}$, a set $\Lambda_{n_j} = \{p \in \mathbb{Z}; -\frac{n_j}{2} < p \leq \frac{n_j}{2}\}$. Then for Fourier coefficients of spline wavelets $\psi_j^{\langle 0,1\rangle}$ in the space $W_j^{\langle 0,1\rangle}$ the relation holds

$$(-1)^{l(d+1)}\widehat{\psi}(p+ln)(p+ln)^{d+1} = \widehat{\psi}(p)p^{d+1}, \quad l \in 2\cdot\mathbb{Z}, \psi \in W_j, p \in \Lambda_{n_j}, \tag{11}$$

where the set $2\cdot\mathbb{Z}$ is the set of even integral numbers.
Proof We can write

$$\psi_j^{\langle 0,1\rangle}(x) = \sum_{p\in\mathbb{Z}} \widehat{\psi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x},$$

where $\widehat{\psi}_j^{\langle 0,1\rangle}(p)$ are Fourier coefficients. It follows from (2)

$$\begin{aligned}\sum_{p\in\mathbb{Z}} \widehat{\psi}_j^{\langle 0,1\rangle}(p) e^{2\pi i p x} &= \psi_j^{\langle 0,1\rangle}(x) \\ &= \sum_{k\in\mathbb{Z}} g_k \phi_{j+1}^{\langle 0,1\rangle}(x - k2^{-j-1}) \\ &= \sum_{k\in\mathbb{Z}} g_k \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{2\pi i p(x-k2^{-j-1})} \\ &= \sum_{p\in\mathbb{Z}} \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) \left(\sum_{k\in\mathbb{Z}} g_k e^{-2\pi i p k 2^{-j-1}}\right) e^{2\pi i p x}.\end{aligned}$$

Now we compare Fourier coefficients.

$$\begin{aligned}
\widehat{\psi}_j^{\langle 0,1\rangle}(p) &= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} g_k e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} (-1)^k h_{1-k} e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} e^{k\pi i} h_{1-k} e^{-2\pi i p k 2^{-j-1}} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p)\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i p(1-k)2^{-j-1}} e^{(1-k)\pi i} \\
&= \widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{\pi i(1-p2^{-j})}\sum_{k\in\mathbb{Z}} h_k e^{2\pi i p k 2^{-j-1}} e^{-k\pi i} \\
&= -\widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{-2\pi i p 2^{-j-1}}\sum_{k\in\mathbb{Z}} h_k e^{-2\pi i k(\frac{1}{2}-p2^{-j-1})} \\
&= -\widehat{\phi}_{j+1}^{\langle 0,1\rangle}(p) e^{-2\pi i p 2^{-j-1}}\overline{m_{j+1}(p-2^j)}
\end{aligned}$$

For $m_{j+1}(p)\neq 0$ we can write

$$\widehat{\psi}_j^{\langle 0,1\rangle}(p) = \widehat{\phi}_j^{\langle 0,1\rangle}(p) m_{j+1}^{-1}(p)\left[-e^{-2\pi i p 2^{-j-1}}\overline{m_{j+1}(p-2^j)}\right]. \tag{12}$$

When we propound $n = 2^j$ and $\Lambda_n = \{p\in\mathbb{Z}; -\frac{n}{2} < p \leq \frac{n}{2}\}$ as in (7) and signify $\widehat{\phi}_j^{\langle 0,1\rangle}(p) \equiv \widehat{\phi}(p)$. For $\Lambda_n = \{p\in\mathbb{Z}; -2^{j-1} < p \leq 2^{j-1}\}$ follows this $m_{j+1}(p)\neq 0$. Further with using the relation (7) we can write

$$\begin{aligned}
&(-1)^{l(d+1)}\widehat{\psi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} = \\
&= (-1)^{l(d+1)}\widehat{\phi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} m_{j+1}^{-1}(p+ln)\cdot \\
&\cdot \left[-e^{\frac{-2\pi i(p+ln)}{2^{j+1}}}\overline{m_{j+1}(p+ln-2^j)}\right].
\end{aligned}$$

Since it holds

$$\begin{aligned}
m_{j+1}(p+ln) &= \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i(p+ln)k2^{-j-1}} \\
&= \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i(p)k2^{-j-1}} e^{-2\pi i(ln)k2^{-j-1}}
\end{aligned}$$

for $n = 2^j$ we obtain

$$m_{j+1}(p+ln) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i(p)k2^{-j-1}}(-1)^{lk}$$

When we write

$$e^{-2\pi i(ln)k2^{-j-1}} = e^{-2\pi ilk}$$

we obtain for even numbers l

$$\begin{aligned}
&(-1)^{l(d+1)}\widehat{\psi}_j^{\langle 0,1\rangle}(p+ln)(p+ln)^{d+1} = \\
&= \widehat{\phi}_j^{\langle 0,1\rangle}(p)p^{d+1}m_{j+1}^{-1}(p)\left[-e^{-2\pi ip2^{-j-1}}\overline{m_{j+1}(p-2^j)}\right] \\
&= \widehat{\psi}_j^{\langle 0,1\rangle}(p)p^{d+1}.
\end{aligned}$$

From preceding it follows the property (11) for spline wavelets.

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