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The Ancient Origin of Symmetry Idea and the Newman Property

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1. Introduction

The vital significance for development of Greek mathematics lied in the idea of an inner similarity – self-reconstruction and self-duplication of a given structures. That way the idea of symmetry was born which was understood by Greeks as a conformity between parts and wholeness as well as between individual parts of wholeness.

Consequently, the idea of similarity is a particular case of the idea of symmetry. Idea of similarity shows invariance of some elements (e.g. the ratio of suitable similar polygon sides) under some kind of transformations. Transformations of space in itself, which preserve its internal structure, are named automorphisms. Hence, this is symmetrical (as a part of a space) what is preserved by automorphisms of the space. Leibniz expressed this thought in philosophical way saying that such two things are symmetrical which we can not diverse when we consider them in itself.

Consider the case of the Pythagorean School. Its vital point was the concept of harmony. We can recognize and create it thanks to numbers and number ratios. The most important aspects of this harmony are: cosmos, analogy, symmetry, eurhythmy, musical harmony and love. Harmony then plays a role of arche and discovered mathematics shows how this world emerged from this harmony. By mathematics we can notice a "unity" between finiteness and infiniteness

Look now at the "golden section" which is the central issue of Pythagorean mathematics.

$$\frac{a+b}{a} = \frac{a}{b}. \tag{20}$$

It was treated as the most perfect kind of harmony. It appoints at special kind of symmetry which can be often found among the living nature. For this reason the Pythagoreans as a symbol of their association adopted a pentagram which was for them a symbol of health and perfect harmony.

Let us introduce the following notation $\phi = \frac{a}{b}$ and $\rho = \phi^{-1} = \frac{b}{a}$. By the fixed proportion we obtain the next formulas: $\phi^2 = \phi + 1$ and $\rho^2 = 1 - \rho$. These formulas allow us to calculate quantities $\phi = \frac{\sqrt{5}+1}{2}$ and $\rho = \frac{\sqrt{5}-1}{2}$. Moreover, we obtain recurrence formulas (by multiplying these formulas by ϕ^n and ρ^n, respectively):

$$\phi^{n+2} = \phi^{n+1} + \phi^n \tag{21}$$

and

$$\rho^{n+2} = \rho^n - \rho^{n+1}. \tag{22}$$

Let $\{u_n\}$ denote the Fibonacci sequence, i.e. a recursively defined sequence as follows:

$$u_1 = u_2 = 1; \tag{4}$$

$$u_{n+2} = u_{n+1} + u_n. \tag{5}$$

Greeks didn't notice a symmetry which lied in infiniteness. As we know, an infinite set is characterized by existing some proper subset which is equinumerous to this whole set. This definition is a realization of the property of inner similarity, i.e. symmetry. In according to Greeks symmetry denotes absolute order and harmony, hence, it should remove, common in that time, the problem of consideration of infinity and chaos as equivalent.

It means that we obtain the following number sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, this is that each next element of the sequence is the sum of the two immediate preceding elements.

Notice that
$$\lim \frac{u_n}{u_{n+1}} = \rho. \qquad (6)$$

In order to prove it is sufficient to divide Eq. (5) by u_{n+1}. We obtain then $\frac{u_{n+2}}{u_{n+1}} = 1 + \frac{u_n}{u_{n+1}}$. Denote $\lim \frac{u_n}{u_{n+1}}$ by r. If $n \to \infty$, then we have $\frac{1}{r} = 1 + r$. The number ρ is the only positive number which fulfils this equation, thus $r = \rho$.

The following relationships are true as well:

$$\phi^{n+1} = \phi u_{n+1} + u_n \qquad (7)$$

and
$$\rho^{n+1} = (-1)^{n+1}(u_n - \rho u_{n+1}). \qquad (8)$$

Let us carry out an inductive proof of the property (7). Trueness of this formula for $n = 1$, i.e. the formula $\phi^2 = \phi + 1$, is caused by formula (2). Assume that for some n we have $\phi^{n+1} = \phi u_{n+1} + u_n$. On the basis of formula (2) and the inductive assumption we obtain $\phi^{n+2} = \phi\phi^{n+1} = \phi(\phi u_{n+1} + u_n) = \phi^2 u_{n+1} + \phi u_n = (\phi+1)u_{n+1} + \phi u_n = \phi u_{n+1} + u_{n+1} + \phi u_n = \phi(u_{n+1} + u_n) + u_{n+1} = \phi u_{n+2} + u_{n+1}$. The last equality is obtained on the basis of the property (5).

That has finished the proof of the property (7). These properties demonstrate the direct relationship between Fibonacci sequence and the geometric sequence ϕ^n. The golden number ϕ is the quotient of this sequence which is created recursively by Fibonacci sequence [6].

The number ϕ appears in many surprising situations. Pythagoreans and their successors had relish for tracing this number in various arrangements and relationships. The most known is that the ratio of a diagonal of a regular pentagon to its side is equal to ϕ, and diagonals intersect at points of the golden section of each of diagonals. Consequently, the points of diagonal intersection create in the pentagon the smaller one. We can repeat this procedure to infinity. The Pythagorean, being fascinating of this relationship, chose the pentagram (i.e. Pythagorean star) as theirs emblem.

It is interesting to notice how different philosophical systems take basic intuitions from the properties of the pentagon, e.g. Plato's cosmology is based on the analysis of Pentad – the number of love, perfect unification, beauty and fertility. It is condensed image of Decade (the half of it) which is the most perfect number expressing the whole Cosmos. Moreover, it is the sum of Dyad (feminine number) and Triad (male number). Then, it is a microcosmos, expresses the nature of human being, and Pentagram is its graphic image.

If we then inscribe a regular decagon within a circle, then the ratio of the radius to the side equals ϕ.

We examine a triangle ABO, whereas O is the center of the circle and A i B are the successive vertices of the regular decagon inscribed within this circle. It is a isosceles triangle with the vertical angle equal to 36°. From the point B we draw an arc with a radius a, which intersects the side OA at a point D_1. As a result we obtain a triangle ABD_1 which is similar to the triangle ABO. It is easily to check that $\frac{a}{R-a} = \frac{R}{a}$, hence $a^2 = R(R-a)$. It is the golden proportion defining the number ϕ.

If we draw from the point D_1 an arc having now $R-a$ as its radius, then this arc intersects the interval AB at a point D_2 and we obtain again a triangle AD_2D_1 similar to the triangle ABO. We can continue this process to infinity and hence we receive an infinite sequence of successive inscribed within itself similar triangles that have a common vertex A. Thus, we obtain a "self-reproducing" structure in the triangle ABO. It is an example of an ideal symmetry obtained in the triangle ABO. We can say that this symmetry causes the gold section.

Pay attention to still one more intriguing property of the golden number. Let $\rho, \rho^2, \rho^3, ...$ be a sequence of numbers on the interval $[0;1]$. We obtain a sequence of intervals

$$I_1 = [\rho^2;\rho], I_2 = [\rho^4;\rho^3], ...I_n = [\rho^{2n};\rho^{2n-1}], ...$$

which have the following property

$$\frac{|I_n|}{\rho^{2n-2}} = \frac{\rho = \rho^2}{1} = \rho(1-\rho) = \rho\rho^2 = \rho^3, \tag{9}$$

i.e. the ratio of intervals lengths I_n to intervals lengths $[0;\rho^{2n-2}]$ is steady and equal to ρ^3. Intervals I_n are situated symmetrically on intervals $[0;\rho^{2n-2}]$, respectively. Moreover, lengths of intervals I_n create a decreasing geometrical sequence with the quotient ρ^2.

Notice that in the case of the classic Cantor set the number $\frac{2}{3}$ is its “generato”, what we mean in the following sense. We take a point symmetrical to the point $\frac{2}{3}$ relative to the middle of $[0;1]$ (this symmetrical point is $\frac{1}{3}$). Then we throw away the inside of the interval $[\frac{1}{3};\frac{2}{3}]$. In the next step we proceed the same with two remained intervals. This procedure is continued to infinity and the rest determines the Cantor set (it is compact, dense in itself, uncountable and has measure zero).

Analogically, each number of the open interval $(\frac{1}{2};1)$ can be treated as a generator of some Cantor set. The measure of this set does not have to be zero, e.g. for the generator $0.6=\frac{3}{5}$ this measure is equal to $\frac{2}{3}$.

Let φ denote a function which for every number from the interval $[\frac{1}{2};1]$ assigns to the measure of a set generated by this number in according to the procedure described above. Obviously, φ is an increasing function and, for instance, $\varphi(\frac{1}{2})=1$, $\varphi(\frac{3}{5})=\frac{2}{3}$, $\varphi(\frac{1}{3})=0$.

Notice that in the case of the analyzed number ρ, which determines the golden section, we have:

$$0.6<\rho<\tfrac{2}{3} \tag{10}$$

and

$$\varphi(\rho)=0. \tag{11}$$

ρ is he smallest number for which the function φ takes the value 0. (12)

Proof of the property (11).

On the successive steps of creating the Cantor set by means of the number ρ we throw out intervals of the lengths:

$|I_1|=\rho-\rho^2=\rho(1-\rho)=\rho\rho^2=\rho^3$;

$2|I_2|=2(\rho^3-\rho^4)=2\rho^3(1-\rho)=2\rho^3\rho^2=2\rho^5$;

$2^{n-1}|I_n|=2^{n-1}(\rho^{2n-1}-\rho^{2n})=2^{n-1}\rho^{2n-1}(1-\rho)=2^{n-1}\rho^{2n-1}\rho^2=2^{n-1}\rho^{2n+1}$ etc.

The sum S of these intervals come to:

$$S=\sum[2^{n-1}\rho^{2n+1}]=\frac{1}{2}\rho\sum[2^n\rho^{2n}]=\frac{1}{2}\rho\sum[(2\rho^2)^n=$$

$$=\frac{1}{2}\rho\frac{2\rho^2}{1-2\rho^2}=\frac{1}{2}\rho\frac{2\rho^2}{1-\rho^2-\rho^2}=$$

$$= \frac{1}{2}\rho\frac{2\rho^2}{\rho-\rho^2} = \frac{1}{2}\rho\frac{2\rho^2}{\rho(1-\rho)} = \frac{1}{2}\rho\frac{2\rho^2}{\rho^3} = 1.$$

This finishes the proof of the property (11).

Proof of the property (12)

For a fixed $k \geq 5, k \in N$ the number $\rho^{2n-1} - \rho^k$ generates some Cantor set. Let $J_{k,n}$ denotes an sequence of intervals

$$[(\rho-\rho^k)\rho^{2(n-1)}; (\rho^2+\rho^k)\rho^{2(n-1)}] = [\rho^{2n}+\rho^k; \rho^{2n-1}-\rho^k].$$

Hence,

$$|J_{k,n}| = (\rho-\rho^k)\rho^{2(n-1)} - (\rho^2+\rho^k)\rho^{2(n-1)} =$$

$$\rho^{2n-1} - \rho^{2n-2+k} - \rho^{2n} - \rho^{2n-2+k} = \rho^{2n-1}(1-\rho) - 2\rho^{2n-2+k} =$$

$$\rho^{2n+1} - 2\rho^{2n-2+k} = \rho^{2n}(\rho - 2\rho^{k-2}).$$

And then the sum of lengths of all throwing out intervals, during the construction of the Cantor set C_k, comes to $1 - 2\rho^{k-3}$, because

$$\sum[J_{k,n}] = \sum[2^{n-1}\rho^{2n}(\rho-2\rho^{k-2}) = (\rho-2\rho^{k-2})\sum[2^{n-1}\rho^{2n} =$$

$$\frac{1}{2}(\rho-2\rho^{k-2})\sum[2^n\rho^{2n} = \frac{1}{2}(\rho-2\rho^{k-2})\sum[(2\rho^2)^n =$$

$$\frac{1}{2}(\rho-2\rho^{k-2})\frac{2\ rho^2}{1-2\rho^2} = \frac{1}{2}(\rho-2\rho^{k-2})\frac{2\rho^2}{\rho^3} = \frac{1}{2}(\rho-2\rho^{k-2})\frac{2}{\rho} = 1-2\rho^{k-3}.$$

Thus, we obtain that the measure of the Cantor set C_k is equal to $2\rho^{k-3}$. If $k \to \infty$, then $|C_k| \to 0$. This also means that for none number $x < \rho$ the measure of the Cantor set generated by x can be equal to zero.

The "maximal" Cantor set of measure zero generated by the number ρ can be named as "the golden Cantor set".

2. Symmetry and the concept of group

Hermann Weyl [13] presumed that every symmetry is connected to some group of transformations. For example, symmetry of a system referring to invariance of passing from one Cartesian system of coordinates to another is connected with a certain group of space transformations, whereas symmetry of a space which is drawn out from undistinguishing of a given system elements (e.g. the set of locations of a given particle) is connected with a group of permutations.

As we noticed before, transformations of a space in itself which preserve its inner structure, are named automorphisms. Symmetry then is connected to invariants of certain automorphisms which form a group of transformations. Consequently, elements preserved by a group of transformations are undistinguishable.

Thus, in Euclidean geometry we treat these figures, which we can transform on each other by means of isometry, as identical. For example, undistinguishable are intervals which have the same length or circles of the same radius. Because all isometries form a group of transformations, so we can treat the Euclidean geometry as some structure based on a group of isometry and investigating invariants of this group. From this observation Felix Klein drew out his proposition to classify diverse geometries on the basis of transformations groups and theirs invariants.

Notice that the geometrical space of the Newton mechanic is affinic space and the invariants are inertial movements preserved by the Galilean transformations [9].

We can say that groups are such tools which recognize symmetrical structures. In the case of the golden section we can find suitable group pointed at some hidden algebraical symmetry as follows.

Notice that the roots of the equation

$$(x^2 - 1)^2 - x^2 = 0$$

are numbers $\rho, -\rho, \phi, -\phi$. On the set of these roots we can define a group of automorphisms $G = \{Id, f_a, f_o\}$, where Id is an identically function, while $f_a(x) = -x$ and $f_o(x) = \frac{1}{x}$. Because $\frac{1}{\rho} = \phi$, it is a permutations group of the above equation roots.

The structure $\{G, *, e\}$ is a group, if: 1. $\bigwedge a, b \in G(a * b \in G)$; 2. $\bigwedge a \in G \bigvee a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$; 3. $\bigwedge a \in G(a * e = e * a = a)$, where e is named the neutral element of the group.

3. Symmetry in the case of topological manifolds

Notice that maps f_a and f_o are involutions, i.e.$f_a^2 = Id$ and $f_o^2 = Id$. It is not accidental because involution play essential role in investigation and recognizing of symmetry. Both rotational symmetry and reflection symmetry (or other kinds of symmetries) are defined by suitable involutions. Obviously, involution together with the identity form a group of transformations.

In 1939 P.A. Smith showed ([11], s. 707-711) that the set of fixed points of a continuous involution defined on a sphere S^n is a sphere of dimension $k < n$, where $n \leq 3$. In the case of the sphere S^3 we have then four possible types of involutions, where the sets of fixed points are respectively: spheres S^2, S^1, S^0 (a pair of points) and S^{-1} (the empty set). It agrees with intuition that sphere symmetries are reduced to mirror symmetries (i.e. plane, axis and centre symmetry). In the case of spheres of dimension $n > 3$ some weaker theorem, saying the set of fixed points of involution creates a homological sphere of lower dimension, is true in Smith-Wilder sense ([11], [2] and [5]).

Analogical theorem is true for n-dimensional balls, $n \leq 4$, because involutions are homeomorphisms, so they transform the boundary of a ball in its boundary and interior in interior.

This perfect symmetry of balls and spheres (little weaker in the case of spheres of dimension $n > 3$) can be partly found in the instance of arbitrary topological manifolds. In 1930 M.H.A. Newman proved the following

Newman's theorem. The set of fixed points of a continuous, non identity involution defined on a connected metric topological manifold is nowhere dense [10].

It means that a subset of manifold, in which we could not "recognize" any symmetry, is very "small".

A homological manifold is a space X, which realizes the following properties:
(i) X is an Euclidean neighborhood retract.
(ii) The singular homology group $H_j(X, X - \{x\}; \mathbb{Z})$ is isomorphic to the group $H_j(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$.

An n-dimensional topological manifold is called a topological space being locally Euclidean n-dimensional space, i.e. for every neighborhood U there exists a compact n-dimensional ball included in the set U.

Therefore, the topological spaces possessing the property expressed by Newman theorem can be called "symmetrical". This property is named as the Newman property, i.e. a space X has **the Newman property** if the set of fixed points of any continuous, non-identity involution on X has empty interior.

Thus, connected manifolds have the Newman property [10].

We can consider the Newman property in a topological space locally. It allows us to see "local symmetries", which often lead to "wholeness" symmetry.

A point x is **a Newman point**([3], p. 1272) if for each neighborhood V of x and for any continuous involution ψ on V the point x is not an accumulation point of the interior of the set of fixed points of ψ as well as of its complement; in other words,

$$x \notin \overline{\operatorname{int} S} \cap \overline{(V - S)}, \tag{13}$$

where S is the set of fixed points of the mentioned involution.

Since manifolds are locally connected and any subregion of a manifold is a connected manifold, we infer that each point of a manifold is the Newman one.

Theorem 1. *Each point of a topological manifold is a Newman point.*

Proof. Assume in contrary that x is not a Newman point. Let $\varphi : V \to V$ be a continuous involution defined in some neighbourhood of x such that the point x is the concentration point of identity-points set interior and its complement. Because manifolds are locally connected, so there is open connected neighbourhood $U \subset V$ of x. Obviously, U has a common part with interior of identity involutions. Because $\varphi x = x$, so the set $W = U \cup \varphi(U)$ is connected. By continuity of the involution φ we can choose the set U so small that W is included in a suitable Euclidean sphere. Thus, φ is a continuous non-identity involution defined on manifold W and the interior of fixed-points set of this involution is non-empty. It is contrary to Newman's theorem.

Later P. A. Smith [12] extended the Newman theorem to homological connected manifolds. The reasoning similar to the one before implies that each point of a homological manifold is the Newman one (cf. Černavskii [4]). Hence, we have

Theorem 2. *If each point of a connected space X is a Newman point then the space X has the Newman property.*

Proof. Assume that X does not have the Newman property. Thus, there is a continuous non-identity involution ψ on X such that the set of fixed points S of ψ has non-empty interior. Fix a point x which belongs to the boundary of the interior of fixed points of the involution ψ. Take any neighbourhood U of x. The involution ψ is an involution on the set $V = U \cap \varphi(U)$ which is a neighbourhood of x.

Since the set of non-fixed points is dense in $X - \operatorname{int} S$, the point x lies in the closure of non-identity points of ψ; moreover, the point x lies in the closure of the set $\operatorname{int} S$. Hence, the point x is not a Newman point. ■

Remark. The converse theorem is not true.

Example. The topological sinusoid $F = \{(x, \sin\frac{1}{x}); x \in (0, 1]\} \cup \{(0, y); y \in [-1, 1], x = 0\}$ has the Newman property, however points of the interval $\{(x, y); y \in [-1, 1], x = 0\}$ are not Newman points.

There exist continua such that all their points are the Newman ones although they are not homological manifolds. Sierpiński universal plane curve is an example.

It wiil mean that this curve has "full" inner symmetry like topological manifolds.

3.1. Involutions on Sierpiński universal curve

Let S be Sierpiński universal curve, i.e. a curve which is constructed of topological disc R by removing the interiors of mutually disjoint topological discs that are contained in the interior of R. These discs must densely fill the disc R. Whyburn [15] showed that continua defined in this way are homeomorphic with one another, particularly with Sierpiński curve constructed of a plane square in the known way. We can assume that boundaries of the components of the complement of S in a plane are mutually disjoint sircles (call this circles *boundary circles of S).* In Sierpiński curve one can mark off two kinds of points:

1. *Rational points*, i.e. points belonging to boundary circles of S.
2. *Irrational points*, i.e. remaining points of S.

Krasinkiewicz [7] proved that *homeomorphism of S on itself maps rational points of S on rational points and irrational points on irrational points.* The method of the proof of this theorem is adapted in the reasoning in the following lemmas.

Lemma 1. *If $h : U \to U$ is a homeomorphism of a region of Sierpiński curve S then boundary circles of S contained in U are mapped on boundary circles of S.*

Proof. Let Z be a boundary circle of S contained in U. The set $h(Z)$ is a simple closed curve in a plane. The curve $h(Z)$ separetes the plane R^2 into two regions R_1 i R_2 and is their common boundary (by Jordan's theorem).

Note that $h(Z)$ is a boundary circle of S because, on the contrary, $U \cap R_1 \neq \emptyset \neq U \cap R_2$. But this contradicts the facts that the property of separating is topological invariant and boundary circles not separate S. ∎

Lemma 2. *1. For any irrational point of S there exist arbitrary small connected neighbourhoods of this point not containing rational points on its boundary.*

2. For any rational point x there exist arbitrary small connected neighbourhoods of this point not containing rational points on its boundary except the points of the boundary circle containing x.

Proof. Let $x \in S$ and let V be any neighbourhood of x.

1. Assume that x is an irrational point of S.

Make the following (upper semi-continuous) decomposition of S. The elements of it are boundary circles being the boundaries of the bounded components of the complement of S in the plane and the points of the unbounded component and all the irrational points of S.

We present the hyperspace K which is topologically equivalent to a closed plane disc (by the Moore theorem [8]). The natural mapping $F : S \to K$ is continuous and monotonic. Since the boundary circles of S is countable number, so a disc K contains only countable number of the points being the images (by the mapping F) of the rational points of S. The number of different circles of the radius less then the fix radius r and with the fix centre is uncountable. Hence there exists circles (contained in K) with the centre at the point $y = F(x)$ and an arbitrary small radius. These circles do not contain the points being the images of the rational points. In the middle of them there exists a circle T such that $F^{-1}(T)$ is contained in V. Since the (continuous) mapping F restricted to the set of the irrational points is the identity, so $F^{-1}(T)$ is a simple closed curve contained in S. Let W denote the geometric interior of the curve $F^{-1}(T)$. The set W is a region of S containing the point x. Its boundary does not contain the rational points of S.

2. If x is a rational point of S, then consider the following (upper semi-continuous) decomposition of S: The elements of it are boundary circles being the boundaries of the bounded components of the complement of S in a plane except the circle T_x including the points x and other points of S.

We present the hyperspace K which is topologically equivalent to a circular plane ring (by the Moore theorem [8]). The natural mapping $F : S \to K$ is continuous and monotonic.

The point $y = F(x)$ is a point of the boundary of K. Let L be an arc lying in the image of the boundary circle T_x and let y lie in the interior of L. There exists an uncountable number of circles that centres lie at the bounded component of the complement of the ring K in the plane. These circles include the ends of L and do not include the images of rational points except the ends of the arc L (the argumentation is similar as in the first part of the proof). Among them there exists a circle T such that $T_0 = F^{-1}((T \cap K) \cup L)$ is included in V. Since the (continuous) mapping F restricted to the sum of the set of irrational points of S and the set T_x is the identity, so T_0 is simple bounded curve in the curve S.

The simple closed curve T_0 does not contain the other rational points of S except the points of the boundary circle containing x . The common part W of the geometric interior of T_0 and S is a region of S. ∎

Theorem 3. *Each point of the Sierpiński plane universal curve is the Newman one.*

Proof. Take an arbitrary point $x \in S$ in the Sierpiński universal curve S. Assume on the contrary that x is not Newman. Thus there exists a neighbourhood V of x and a continuous involution ψ on V

that satisfy the condition

$$x \in \overline{\operatorname{int} Z} \cap \overline{(V - Z)}, \tag{14}$$

where Z is the set of the fixed points of the mentioned involution.

Since the curve S is locally connected, we may assume that the neighbourhood V is connected.

Consider two cases:

1. The point x is an irrational point of S. By lemma 2 (part 1) there exists connected neighbourhood W of x contained in V together with its closure which does not contain the rational points of S on its boundary. So, all the boundary circles of S either are contained in W or are disjointed with this set. Moreover, the involution φ maps boundary circles contained in W onto boundary circles.

Each point y of a disc can be uniquely represented as a convex combination $y = ty_0 + (1-t)y_1, 0 \leq t \leq 1$, of the centre y_0 of the disc and the point y_1 on its boundary. Let C be an arbitrary boundary circle of S contained in W. Extend $\varphi|W$ from the circle C to the involution φ^* defined on the disc Q_C (the circle C is the boundary of Q_C) in the following way: If $y \in Q_C$, then $\varphi^*(y) = ty_0' + (1-t)\varphi(y_1)$, where y_0' is the centre of the circle $\varphi(C)$.

In this way we give an imbedding $\varphi^* W^* \to R^2$ defined on the region W^* of the plane such that $W^* \cap S = W$. Note that φ^* is the identity on these discs and that the function φ is the identity on the boundaries of them; there exist discs such that the function φ^* is not the identity on them too. Consider the set $U^* = W^* \cup \varphi^*(W^*)$. The set U^* is the region of a plane (by the Brouwer property of a plane) and φ^* is an involution on it.

If we assume that the set of fixed points of the involution φ on V has non-empty interior, then the set of fixed points of the involution φ^* extended to W^* has non-empty interior too; but by Newman theorem it is impossible.

2. Let x be a rational point of S. By lemma 2 (part 2) there exists connected neighbourhood W of x contained in V together with its closure which does not contain the rational points of S on its boundary except the points of the boundary circle T_x containing x. So, the boundary of the set $W_1 = W \setminus T_x$, being a region of S, consists of irrational points of S.

So, all the boundary circles of S either are contained in W_1 or are disjointed with this set. Moreover (by lemma 1) the involution φ maps boundary circles contained in W_1 onto boundary circles.

Analogically, as at first part of the proof, we extend the function $\varphi|W_1$ to function φ^* defined on the region W_1^* of a plane. In this way we give an imbedding $\varphi^* W_1^* \to R^2$ defined on the region W_1^* of the plane such that $W_1^* \cap S = W_1$. Note that φ^* is the identity on these discs and that the function φ is the identity on the boundaries of them; there exist discs such that the function φ^* is not the identity on them too. Consider the set $U^* = W_1^* \cup \varphi^*(W_1^*)$. The set U^* is the region of a plane (by the Brouwer property of a plane) and φ^* is an involution on it.

The point x lies in the interior of W and is a condensation point of the interior of the fixed points as well as of the non-fixed points of φ. Since the set T_x is boundary in the curve S, so $W_1 \cap \operatorname{int} Z \neq \emptyset \neq W_1 \cap (V \setminus Z)$. Hence can see that involution φ^* (defined on the region $W_1^* \cup \varphi^*(W_1^*)$ of the plane) is non-identity involution and the set of the fixed points of its has the non-empty interior. But this contradicts the Newman theorem.

Note that points of Menger spatial curve are not Newman (it is followed from Anderson characterization theorem of Menger spatial curve)[1].

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