Jan Długosz University of Częstochowa
Scientific Issues, Mathematics XII, Częstochowa 2007

APPLYING THE IDEA OF FUSIONISM IN THE PROBABILITY THEORY

Maciej Major

Institute of Mathematics, Pedagogical University of Cracov
ul. Podchorążych 2, 30-084 Kraków, Poland
e-mail: mmajor@ap.krakow.pl

Abstract. The solutions presented in this paper may serve as an illustration of „the principle of internal integration", know as the idea of fusionism. In the paper we consider some problem. From an urn containing b white balls and c black ones are selected simultaneously some balls. If the balls are of the same colours one of the players wins, otherwise the other player is the winner. For which values of b and c is the game fair?

Let us consider an urn containing b white balls and c black balls. Let us assume, k balls ($k \geq 2$) are selected simultaneously from our urn. If both balls are of the same colour, then one of the players wins and if the balls are of different colours, then the other player is the winner. For what values of b and c is this game fair?

Some solutions of this problem for $k = 2$ are presented in [2].

In the present paper we suggest three other solutions.

The conditions of the problem imply that $b \geq k$ and $c \geq 1$ or $b \geq 1$ and $c \geq k$. Let us treat all the white balls and all the black ones as distinct objects. Under such assumptions the outcome of such an experiment is a combination of k elements out of the set of $b + c$ balls and the model of this experiment is a classic sample space (Ω, p).

Let us consider the following events:

$A = \{$*both selected balls are of the same colour*$\}$,

$B = \{$*the selected balls are of different colours*$\}$.

Solution problem for $k = 2$.

Therefore

$$P(A) = \frac{b(b-1) + c(c-1)}{(b+c)(b+c-1)},$$

$$P(B) = \frac{2bc}{(b+c)(b+c-1)}.$$

In the sample space (Ω, p) the system $\{A, B\}$ is a complete system of events and therefore, the game is fair if the following condition holds:

$$P(A) = 1/2.$$

These condition is equivalent to the condition

$$b^2 + c^2 - b - c - 2bc = 0. \tag{1}$$

Let us consider the equation

$$x^2 + y^2 - x - y - 2xy = 0, \tag{2}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

It means that equation (2) describes the curve which is symmetrical to the straight line $y = x$.

We can find the natural solutions of the equation (2). It can be easily verified that pairs $(0, 0)$, $(1, 0)$, $(1, 0)$ satisfy equation (2). The process of finding consecutive natural solutions of the equations (2) is presented in Fig. 1.

Solution of the problem for $k = 3$.

Therefore

$$P(A) = \frac{b^3 - 3b^2 + 2b + 2c - 3c^2 + c^3}{b^3 - 3b^2 + 2b + 3b^2c - 6bc + 3bc^2 + 2c - 3c^2 + c^3}.$$

The game is fair if:

$$P(A) = 1/2.$$

This condition is equivalent to the condition

$$b^3 - 3b^2 + 2b - 3b^2c + 6bc - 3bc^2 + 2c - 3c^2 + c^3 = 0. \tag{3}$$

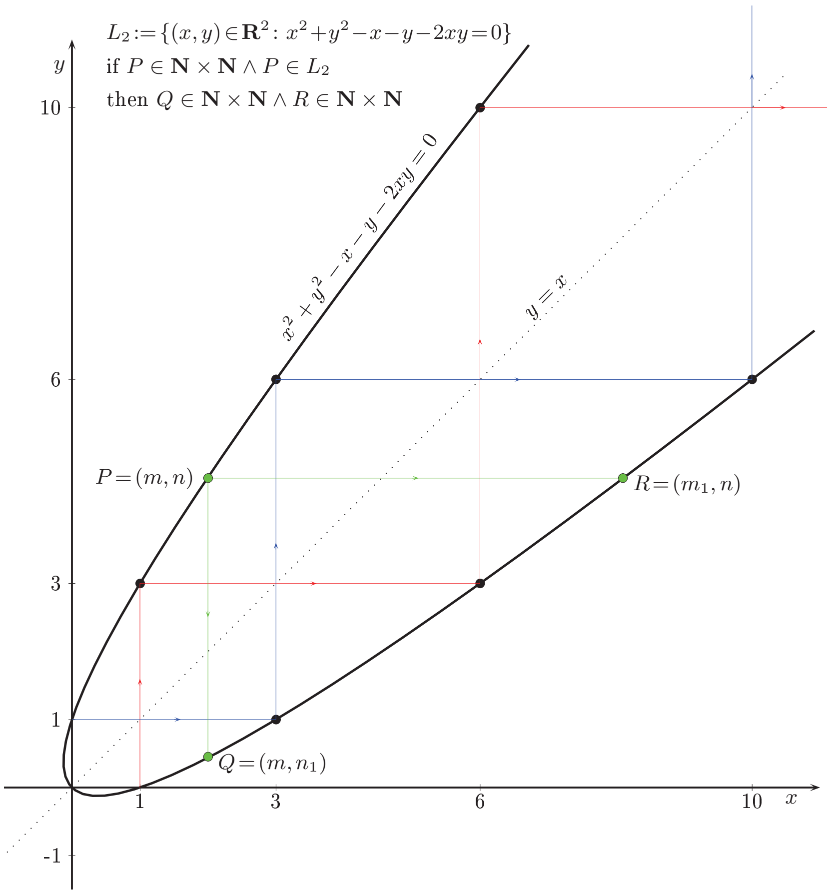

Figure 1:

Let us consider the equation

$$x^3 - 3x^2 + 2x - 3x^2y + 6xy - 3xy^2 + 2y - 3y^2 + y^3 = 0, \tag{4}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

Thus condition (4) and

$$(y + x - 2)(x^2 - x - 4xy - y - y^2) = 0 \tag{5}$$

are equivalent.

Fig. 2 shows the curves representing condition 4 (see [3], pp. 133-138).

We can find the natural solutions of the equation (5). It can be easily verified that pairs $(0, 0)$, $(1, 0)$, $(1, 0)$ satisfy equation (5). The process of finding consecutive natural solutions of the equations (5) is presented in Fig. 3.

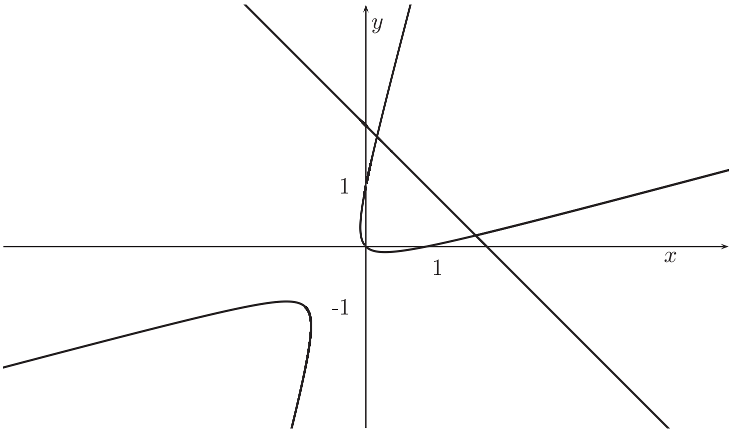

Figure 2:

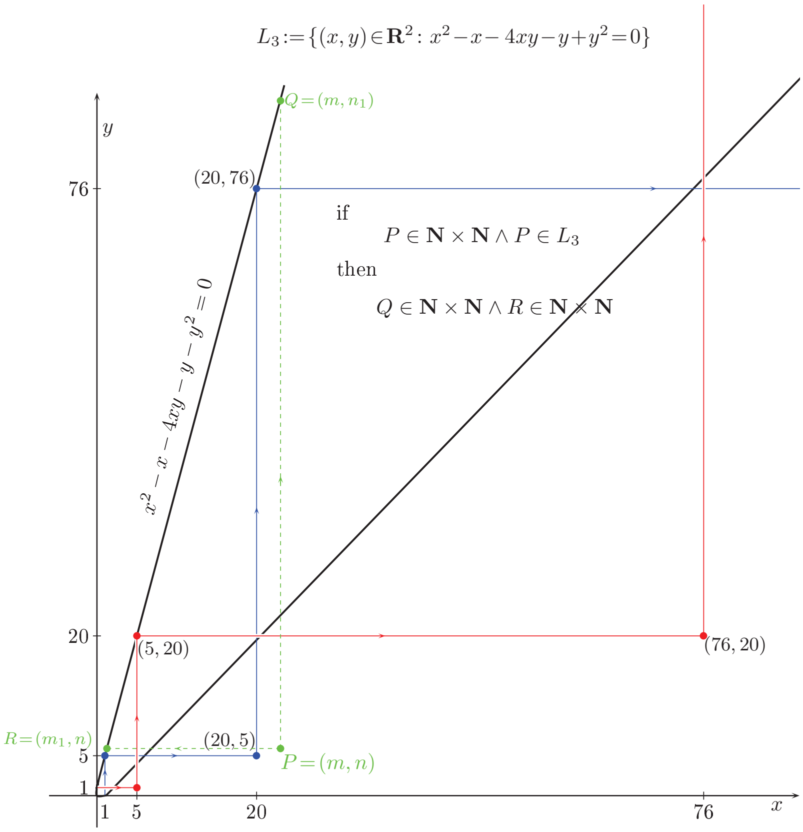

Figure 3:

Some remarks for $k \geq 2$.

Let $k \geq 4$.

Because the game is fair if $P(A) = \frac{1}{2}$ then

$$2[b(b-1)\cdot\ldots\cdot(b-k+1) + c(c-1)\cdot\ldots\cdot(c-k+1)] \\ -(b+c)(b+c-1)\cdot\ldots\cdot(b+c-k+1) = 0. \tag{6}$$

Let us consider the equation

$$2[x(x-1)\cdot\ldots\cdot(x-k+1) + y(y-1)\cdot\ldots\cdot(y-k+1)] \\ -(x+y)(x+y-1)\cdot\ldots\cdot(x+y-k+1) = 0. \tag{7}$$

where $x \in \mathbf{R}$ i $y \in \mathbf{R}$.

Let

$$L_k = \{(x,y) \in \mathbf{R}^2 \colon 2[x(x-1)\cdot\ldots\cdot(x-k+1) + y(y-1)\cdot\ldots\cdot(y-k+1)] \\ -(x+y)(x+y-1)\cdot\ldots\cdot(x+y-k+1) = 0$$

for $k = 2, 3, 4, \ldots$

Figures 4-6 shows the curves representing condition (7) for $k = 4$, 5, 6 respectively.

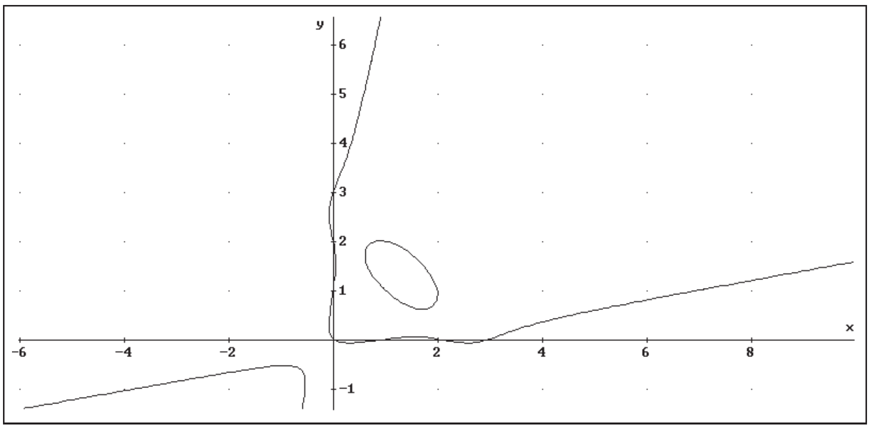

Figure 4:

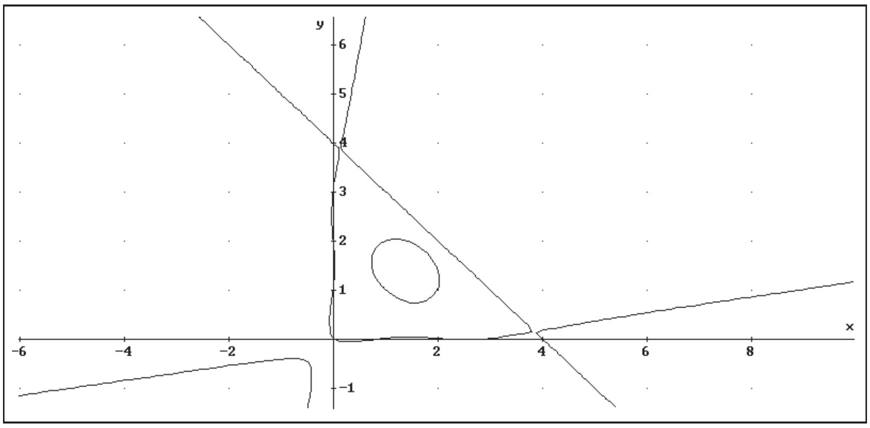

Figure 5:

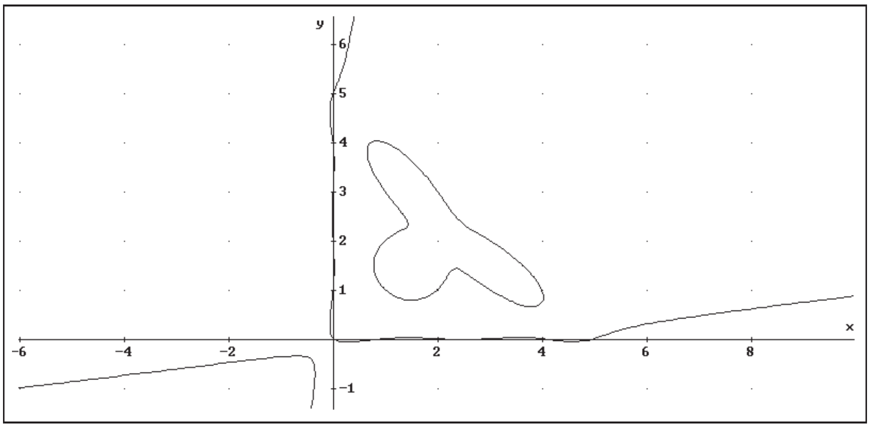

Figure 6:

In the paper a method of finding the solutions of (1) and (3) was given. Unfortunately, this method does not give the complete solution of (6).

It can be easily verified that pairs $(0, 0)$, $(1, 0)$, $(1, 0)$ satisfy equation (7).

Applying a similar reasoning to this used for finding the solutions of (2) and (4) yields

– points $(1, 7)$ and $(7, 1)$ belong to L_4,

– points $(1, 9)$ and $(9, 1)$ belong to L_5,

– points $(1, 11)$ and $(11, 1)$ belong to L_6.

Obtained results lead to the following conclusion. The game is fair if the urn contains 1 white ball and $2k - 1$ black balls (or $2k - 1$ white balls and 1 black ball).

We shall prove the above remark.

If the urn contains 1 white ball and b black balls, then the condition

$P(A) = P(B)$ is equivalent to

$$\binom{b}{k} = \binom{b}{k-1}.$$

It follows that

$$b = 2k - 1.$$

Some generalization of the above game

Now consider a generalization of the above game.

Let us consider an urn containing b white balls, c black balls and z green bals. Assume that 2 balls are selected simultaneously from our urn. If both balls are of the same colour, then one of the players wins and if the balls are of different colours, then the other player is the winner. For what values of b, c and z is this game fair?

The conditions of the problem imply that ($b \geq 2$ and $c \geq 1$ and $z \geq 1$) or ($b \geq 1$ and $c \geq 2$ and $z \geq 1$) or ($b \geq 1$ and $c \geq 1$ and $z \geq 2$). Let us treat all the white balls and all the black balls and all the green ones as distinct objects. Under such assumptions the outcome of such an experiment is a combination of 2 elements out of the set of $b + c + z$ balls and the model of this experiment is a classic sample space (Ω, p).

Let us consider the following events:

$A = \{$*both selected balls are of the same colour*$\}$,

$B = \{$*the selected balls are of different colours*$\}$.

Therefore

$$P(A) = \frac{b(b-1) + c(c-1) + z(z-1)}{(b+c+z)(b+c+z-1)}.$$

The game is fair if the following condition holds:

$$P(A) = \frac{1}{2}.$$

This condition is equivalent to the condition

$$b^2 + c^2 + z^2 - b - c - z - 2bc - 2cz - 2bz = 0. \tag{8}$$

Let us consider the equation

$$x^2 + y^2 + z^2 - x - y - z - 2xy - 2yz - 2xz = 0, \tag{9}$$

where $x \in \mathbf{R}$, $y \in \mathbf{R}$, $z \in \mathbf{R}$.

It means that the equation (9) describes the paraboloida.

In the paper [2] it was proved that all solutions of (1) are of the form

$$b = \frac{k^2+k}{2}, \quad c = \frac{k^2-k}{2}, \quad \text{for } k \in \mathbf{Z} \setminus \{-1, 0, 1\}.$$

Substituting these equalities to (8) we get

$$z = 2k^2 + 1.$$

Hence, the game is fair if:
$b = \frac{k^2+k}{2}, \quad c = \frac{k^2-k}{2}, \quad z = 2k^2+1$; or
$b = \frac{k^2-k}{2}, \quad c = 2k^2+1, \quad z = \frac{k^2+k}{2}$; or
$b = 2k^2+1, \quad c = \frac{k^2+k}{2}, \quad z = \frac{k^2-k}{2}$
for $k \in \mathbf{Z} \setminus \{-1, 0, 1\}$.

The solutions presented above may serve as an illustration of "the principle of internal integration", known as the idea of fusionism. According to this principle, the process of teaching various modules of the school curriculum on mathematics should be conducted so that they could support one another and play a certain role in one another's creation (see [3], p. 39). Particular sections of mathematics appear in the process of teaching as separate threads. At various stages of this process these threads may be linked in order to create a certain unity. Questions offered by the problems on probability may serve as a considerable source of such opportunities.

References

[1] R. Duda, *Zasada paralelizmu w dydaktyce*, Dydaktyka Matematyki, 1, 1982.

[2] M. Major, *The idea of fusionism in the probability theory, based on different solutions to a certain problem*, in: Acta Universitatis Purkyniane 72, Studia Mathematica, Czech-Polish Mathematical School 2001, Ústi nad Labem, pp. 29-133, 2001.

[3] M. Major, *Solutions to a certain problem illustrating the idea of fusionism in the probability theory*, in: Annales Academiae Paedagogice Cracoviensis, 16, Studia Mathematica III, Wydawnictwo Naukowe Akademii Pedagogicznej, Kraków, pp. 133-138, 2003.

[4] A. Płocki, *Stochastyka 1. Rachunek prawdopodobieśtwa i statystyka matematyczna „in statu nascendi"*, Wydawnictwo Naukowe WSP, Kraków, 1997.

[5] A. Płocki, *Stochastyka 2. Rachunek prawdopodobieśtwa i statystyka matematyczna, „zarys dydaktyki*, Wydawnictwo Naukowe WSP, Kraków, 1997.

[6] G. Polya, *Jak to rozwiązać*, PWN, Warszawa, 1964.