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PROBLEM OF THE EXISTENCE OF $\omega^\star$-PRIMITIVES

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Abstract. If (X, ϱ) is a dense in itself metric space and $f\colon X \to \mathbb{R}$, then we define $\omega^*(f,x) = \inf_{r>0} \sup_{y,z\in \mathbf{B}(x,r)\setminus\{x\}} |f(y)-f(z)|$. We say that a function $F\colon X \to \mathbb{R}$ is an $\omega^\star$- primitive for $f\colon X \to \mathbb{R}$ if $\omega^\star(F,\cdot) = f$. We discuss problem of the existence of $\omega^\star$-primitives for an arbitrary upper semicontinuous function $f\colon X \to [0,\infty)$ defined on a dense in itself metric space. At the end we show that if an upper semicontinuous function $f\colon X \to [0,\infty)$ is defined on a nonmetrizable topological space, then $\omega^\star$-primitive may not exists.

Let (X, ϱ) be a metric space, $\mathbf{B}(x,r)$ is an open ball with center x and radius r and let $f\colon X \to \mathbb{R}$ be any function. Then we may define an oscillation of the function f as:

$$\omega(f,x) = \inf_{r>0} \sup_{y,z\in \mathbf{B}(x,r)} |f(y)-f(z)|.$$

It is well known that $\omega(f,\cdot) : X \to [0,+\infty]$ is an upper semicontinuous function vanishing at isolated points of X. There were investigate the following problem.

Problem 1. Let $f\colon X \to [0,+\infty]$ be any upper semicontinuous function vanishing at isolated points of X. Does there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$?

Positive answer was given by profesor J. Ewert and profesor S. Ponomarev:

Theorem 7 ([?]). Let (X, ϱ) be an arbitrary metric space. For every upper semicontinuous function $f\colon X \to [0,+\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F,\cdot) = f$.

In the paper we consider similar problem. Let (X, ϱ) be a dense in itself metric space and let $f\colon X \to \mathbb{R}$ be any function. Then we may define a function $\omega^*(f,\cdot)\colon X \to [0,+\infty]$,

$$\omega^*(f,x) = \inf_{r>0} \sup_{y,z\in \mathbf{B}(x,r)\setminus\{x\}} |f(y) - f(z)|.$$

Similarly, $\omega(f,\cdot)$ is an upper semicontinuous function. Although, definitions of $\omega(f,\cdot)$ and $\omega^*(f,\cdot)$ are similar, their properties may be quite different.

Example 1. Let $X = \{\frac{2k-1}{2^n} : k = 1,\dots, 2^{n-1}, n \geq 0\} \subset \mathbb{R}$ and $f\colon X \to \mathbb{R}$, $f(\frac{2k-1}{2^n}) = \frac{1}{2^n}$ for $\frac{2k-1}{2^n} \in X$.

It is easily seen that $\omega(f,\cdot) = f$ and $\omega^*(f,\cdot) = 0$. Hence, $\omega(f,x) \neq \omega^*(f,x)$ for $x \in X$.

So, we have the following question:

Problem 2. Let (X, ϱ) be a dense in itself metric space and let $f\colon X \to [0,+\infty]$ be an upper semicontinuous function. Does there exists a function $F\colon X \to \mathbb{R}$ such that $\omega^*(F,\cdot) = f$?

We say that a function $F\colon X \to \mathbb{R}$ is an ω^*- primitive for $f\colon X \to \mathbb{R}$ if $\omega^\star(F,\cdot) = f$.

First, we make some observations. For a function $F\colon X \to \mathbb{R}$ we may define upper and lower Baire functions:

$$M_f(x) = \inf_{r>0} \sup_{y\in \mathbf{B}(x,r)} f(y)$$

and

$$m_f(x) = \sup_{r>0} \inf_{y\in \mathbf{B}(x,r)} f(y).$$

Then $\omega(F,x) = M_f(x) - m_f(x)$ for $x \in X$.

Next, if (X, ϱ) is a dense in itself metric space then for a function $F\colon X \to \mathbb{R}$ we may define

$$\limsup_{t\to x} f(t) = \inf_{r>0} \sup_{y\in \mathbf{B}(x,r)\setminus\{x\}} f(y),$$

$$\liminf_{t\to x} f(t) = \sup_{r>0} \inf_{y\in \mathbf{B}(x,r)\setminus\{x\}} f(y)$$

and then

$$\omega^*(F,x) = \limsup_{t\to x} f(t) - \liminf_{t\to x} f(t)$$

for $x \in X$ (if we assume that $\infty - \infty = \infty = -\infty - (-\infty)$)

In the following we will need the following denotations. Let $\varrho(x,A) = \inf\{\varrho(x,a) : a \in A\}$ denotes the distance of the point x from the nonempty set A in a metric space (X,ϱ) and let

$$\mathbf{B}(A,\varepsilon) = \bigcup_{x\in A}\{t \in X : d(x,t) < \varepsilon\} = \bigcup_{x\in A}\mathbf{B}(x,\varepsilon).$$

for $\emptyset \neq A \subset X$ and $\varepsilon > 0$.

We will give the positive answer of Problem 2 in the case of upper semi-continuous functions with finite values $f\colon X \to [0,+\infty)$. We can prove even more. First, we start from the following technical lemma.

Lemma 1. Let (X,ϱ) be a metric space. For every subset M dense in X, nonempty set $A \subset X$ and $\varepsilon > 0$ there exists a set $T_{M,A,\varepsilon} \subset M$ such that

[1] $\varrho(z_1,z_2) \geq \varepsilon$ for every $z_1,z_2 \in T_{M,A,\varepsilon}$,

[2] $\varrho(z,A) < \varepsilon$ for every $z \in T_{M,A,\varepsilon}$,

[3] $\varrho(x,T_{M,A,\varepsilon}) < 2\varepsilon$ for every $x \in A$.

Proof. Observe that another way of stating (2) is to say $T_{M,A,\varepsilon} \subset \mathbf{B}(A,\varepsilon)$ and an equivalent formulation of (3) is $A \subset \mathbf{B}(T_{M,A,\varepsilon},2\varepsilon)$. Since M is a dense subset of X, $M \cap \mathbf{B}(A,\varepsilon) \neq \emptyset$.

Let $\mathfrak{B}$ be the set of all subsets B of X satisfying the following conditions

(a) $B \subset M \cap \mathbf{B}(A,\varepsilon)$,

(b) $\varrho(z_1,z_2) \geq \varepsilon$ for each $z_1,z_2 \in B$.

The family $\mathfrak{B}$ is nonempty because contains all singletons $\{x\}$ for $x \in M \cap \mathbf{B}(A,\varepsilon)$. Moreover, $\mathfrak{B}$ is partially ordered by inclusion. It is easily seen that if $\{B_s : s \in S\}$ is a chain in X then the set $B = \bigcup_{s\in S} B_s$ belongs to $\mathfrak{B}$ and B is above all elements from $\{B_s : s \in S\}$. Hence, by Zorn Lemma the family $\mathfrak{B}$ has a maximal element $T_{M,A,\varepsilon}$.

We will show that the set $T_{M,A,\varepsilon}$ fulfils all required properties. By (a) it is clear that $T_{M,A,\varepsilon} \subset M$ and $T_{M,A,\varepsilon} \subset \mathbf{B}(A,\varepsilon)$, so $\varrho(z,A) < \varepsilon$ for every $z \in T_{M,A,\varepsilon}$. Next $\varrho(z_1,z_2) \geq \varepsilon$ for $z_1,z_2 \in T_{M,A,\varepsilon}$ from (b).

Assume that $\varrho(x_0,T_{M,A,\varepsilon}) \geq 2\varepsilon$ for some $x_0 \in A$. Since M is a dense subset of X, there exists $z_0 \in M$ such that $\varrho(x_0,z_0) < \varepsilon$. Hence

$$\varrho(t,z_0) \geq \varrho(t,x_0) - \varrho(x_0,z_0) \geq \varrho(x_0,T_{M,A,\varepsilon}) - \varrho(x_0,z_0) > 2\varepsilon - \varepsilon = \varepsilon$$

for each $t \in T_{M,A,\varepsilon}$. It follows that $T_{M,A,\varepsilon} \cup \{z_0\} \in \mathfrak{B}$. Since $T_{M,A,\varepsilon}$ is a maximal element of $\mathfrak{B}$, this is a contradiction. Therefore $\varrho(x,T_{M,A,\varepsilon}) < 2\varepsilon$ for every $x \in A$ and the set $T_{M,A,\varepsilon}$ satisfies conditions (1) – (3). $\square$

Remark 1. From condition (1) of the Lemma it follows that $T_{M,A,\varepsilon}$ is a closed and discrete set.

Now, we formulate the main theorem of the paper

Theorem 8. Let (X, ϱ) be a dense in itself metric space and let Y be dense subset of X. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be a pair of functions such that f is upper semicontinuous, g is lower semicontinuous and $g \leq f$. Then there exists **one** function $F\colon X \to \mathbb{R}$ for which

[1] $\limsup_{t\to x} F(t) = f(x)$ and $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$,

[2] $F(x) = g(x)$ for $x \in X \setminus Y$.

Proof. Let

$$K = \{(n, k) \in \mathbb{Z} : -n^2 \leq k < n^2\}.$$

Let $\preceq$ be a relation in K defined as follows

$$(n_1, k_1) \preceq (n_2, k_2) \Leftrightarrow n_1 < n_2 \vee (n_1 = n_2 \wedge k_1 \leq k_2).$$

It is easily seen that K is well ordered by $\preceq$. Define

$$A_{n,k} = \left\{x \in X : \tfrac{k}{n} \leq f(x) < \tfrac{k+1}{n}\right\}$$

and

$$B_{n,k} = \left\{x \in X : \tfrac{k}{n} \leq g(x) < \tfrac{k+1}{n}\right\}$$

for $(n, k) \in K$. We shall construct two families $\{R_{n,k} : (n, k) \in K\}$ and $\{S_{n,k} : (n, k) \in K\}$ of closed and discrete subsets of X which satisfy the following conditions:

(a) $R_{n_1,k_1} \cap R_{n_2,k_2} = \emptyset = S_{n_1,k_1} \cap S_{n_2,k_2}$ for $(n_1, k_1), (n_2, k_2) \in K$, $(n_1, k_1) \neq (n_2, k_2)$ and $R_{n,k} \cap S_{i,j} = \emptyset$ for $(n, k), (i, j) \in K$,

(b) $\bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}) \subset Y$,

(c) $R_{n,k} \subset \mathbf{B}(A_{n,k}, \frac{1}{n})$, $S_{n,k} \subset \mathbf{B}(B_{n,k}, \frac{1}{n})$ for $(n, k) \in K$,

(d) $\varrho(x, R_{n,k}) < \frac{2}{n}$ for $x \in A_{n,k}$, $(n, k) \in K$ and $\varrho(x, S_{n,k}) < \frac{2}{n}$ for $x \in B_{n,k}$, $(n, k) \in K$.

If $(n,k) \in K$ and $A_{n,k} = \emptyset$ then we set $R_{n,k} = \emptyset$ and if $B_{n,k} = \emptyset$ then we set $S_{n,k} = \emptyset$. Thus we have to define $R_{n,k}$ if $A_{n,k} \neq \emptyset$ and $S_{n,k}$ if $B_{n,k} \neq \emptyset$. We will make it inductively. Let $R_{1,-1} = T_{Y,A_{1,-1},1}$ where $T_{Y,A_{1,-1},1}$ is the set from Lemma 1 for $M = Y$, $A = A_{1,-1}$ and $\varepsilon = 1$. Since $R_{1,-1}$ is a closed and discrete subset of X and X is dense in itself, the set $Y \setminus R_{1,-1}$ is dense in X. Thus we can set $S_{1,-1} = T_{Y\setminus R_{1,-1},B_{1,-1},1}$. Next, let

$$\widetilde{Y}_{1,0} = Y \setminus (R_{1,-1} \cup S_{1,-1}), \quad R_{1,0} = T_{\widetilde{Y}_{1,0},A_{1,0},1} \text{ and } S_{1,0} = T_{\widetilde{Y}_{1,0}\setminus R_{1,0},B_{1,0},1}.$$

Fix $(n,k) \in K$. Assume that the closed and discrete sets $R_{i,j}$ and $S_{i,j}$ satisfying conditions (a)-(d) are choosen for $(i,j) \prec (n,k)$ and let

$$\widetilde{Y}_{n,k} = Y \setminus \textstyle\bigcup_{(i,j)\prec(n,k)}(R_{i,j} \cup S_{i,j}).$$

Define $$R_{n,k} = T_{\widetilde{Y}_{n,k},A_{n,k},\frac{1}{n}} \quad \text{and} \quad S_{n,k} = T_{\widetilde{Y}_{n,k}\setminus R_{n,k},B_{n,k},\frac{1}{n}}.$$

It is obvious that the families

$$\{R_{n,k} : (n,k) \in K\} \quad \text{i} \quad \{S_{n,k} : (n,k) \in K\}$$

constructed inductively satisfy conditions $(a) - (d)$. Let us define a function $F\colon X \to \mathbb{R}$ as follows

$$F(x) = \begin{cases} \frac{k}{n} & if \quad x \in R_{n,k}, \quad (n,k) \in K, \\ \frac{k+1}{n} & if \quad x \in S_{n,k}, \quad (n,k) \in K, \\ g(x) & if \quad x \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}). \end{cases}$$

We shall show that (1) and (2) hold. Fix $x_0 \in X$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$ and $f(x_0) < n_0 + 1$. For every $n \geq n_0$ we may find $-n^2 \leq k_n < n^2$ for which $\frac{k_n}{n_0} \leq f(x_0) < \frac{k_n+1}{n_0}$. Thus $x_0 \in A_{n,k_n}$. From (d) for every $n \geq n_0$ there exists $y_n \in R_{n,k_n}$ such that $d(x_0, y_n) < \frac{2}{n}$. Hence $\lim_{n\to\infty} y_n = x_0$. From this we obtain

$$F(y_n) = \frac{k_n}{n} \quad \text{and} \quad 0 \leq f(x) - F(y_n) < \frac{1}{n}.$$

This gives $\lim_{n\to\infty} F(y_n) = f(x_0)$. Thus we have proved that

$$(*) \qquad \limsup_{x\to x_0} f(x) \geq f(x_0).$$

In the same manner we can see that $\liminf_{x\to x_0} f(x) \leq g(x_0)$.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of elements of X converging to x_0, $x_m \neq x_0$ for $n \in \mathbb{N}$ and $\lim_{m\to\infty} F(x_m) = \alpha$, $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$. Without the loss of generality we may assume that all elements of the sequence belong to one of the three sets

$$\bigcup_{(n,k)\in K} R_{n,k}, \quad \bigcup_{(n,k)\in K} S_{n,k} \quad \text{or} \quad X \setminus \bigcup_{(n,k)\in K} (R_{n,k} \cup S_{n,k}).$$

First, suppose that $x_m \in \bigcup_{(n,k)\in K} R_{n,k}$ for $m \geq 1$. Then for every $m \in \mathbb{N}$ we can find $(n_m, k_m) \in K$ such that $x_m \in R_{n_m,k_m}$. The sets $R_{n,k}$ are closed and discrete and for fixed $m \in \mathbb{N}$ there is only a finite number $k \in \mathbb{Z}$ for which $(n,k) \in K$. Besides, $(x_m)_{m\in\mathbb{N}}$ is convergent and is not constant. Hence $\lim_{m\to\infty} n_m = +\infty$. From (c) for every $m \in \mathbb{N}$ there exists $z_m \in A_{n_m,k_m}$ such that $d(x_m, z_m) < \frac{2}{n}$. Moreover

$$F(x_m) = \frac{k_m}{n_m} \quad \text{and} \quad \frac{k_m}{n_m} \le f(z_m) < \frac{k_m+1}{n_m}.$$

Since the function f is upper semicontinuous,

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} f(z_m) \le f(x_0).$$

Now, let $x_m \in \bigcup_{(n,k)\in K} S_{n,k}$ for $m \ge 1$. Then for every $m \in \mathbb{N}$ we can find $(n_m, k_m) \in K$ such that $x_m \in S_{n_m,k_m}$. In the same manner as before we can prove that $\lim_{m\to\infty} n_m = +\infty$. From (c) for every $m \in \mathbb{N}$ there exists $z_m \in B_{n_m,k_m}$ such that $d(x_m, z_m) < \frac{2}{n}$. Besides

$$F(x_m) = \frac{k_m+1}{n_m} \quad \text{and} \quad \frac{k_m}{n_m} \le g(z_m) < \frac{k_m+1}{n_m}.$$

Since $g \le f$ and f is upper semicontinuous, it follows that

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(z_m) \le \limsup_{m\to\infty} f(z_m) \le f(x_0).$$

.

At the end, if $x_m \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$, then $F(x_m) = g(x_m)$ for $m \in \mathbb{N}$. Therefore

$$\alpha = \lim_{m\to\infty} F(x_m) = \lim_{m\to\infty} g(x_m) \le \limsup_{m\to\infty} f(x_m) \le f(x_0).$$

Thus we have proved that $\alpha \le f(x_0)$. Since α is an arbitrary limit number of f at x_0, $\limsup_{x\to x_0} F(x) \le f(x_0)$. Together, with $(*)$ we get

$$\limsup_{x\to x_0} F(x) = f(x_0)$$

for every $x_0 \in X$.

Applying lower semicontinuity of g in the same way we can prove $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$. The equality $F(x) = g(x)$ for $x \in X \setminus Y$ is obvious, becouse $\bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}) \subset Y$ and $F(x) = g(x)$ for $x \notin \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k})$. The proof is complete. □

Remark 2. If under the notation from the proof of the last theorem we define a function $\widetilde{F}\colon X \to \mathbb{R}$ in the following way

$$\widetilde{F}(x) = \begin{cases} \frac{k}{n} & if \quad x \in R_{n,k}, \quad (n,k) \in K, \\ \frac{k+1}{n} & if \quad x \in S_{n,k}, \quad (n,k) \in K, \\ f(x) & if \quad x \in X \setminus \bigcup_{(n,k)\in K}(R_{n,k} \cup S_{n,k}), \end{cases}$$

then it is easily seen that

$$\limsup_{t\to x} \widetilde{F}(t) = f(x) \text{ and } \liminf_{t\to x} \widetilde{F}(t) = g(x) \text{ for } x \in X.$$

Hence we get a theorem analogous with Theorem 8.

Theorem 9. Let (X, ϱ) be a dense in itself metric space and let Y be dense subset of X. Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be a pair of functions such that f is upper semicontinuous, g is lower semicontinuous and $g \le f$. Then there exists a function $F\colon X \to \mathbb{R}$ for which

[1] $\limsup_{t\to x} F(t) = f(x)$ and $\liminf_{t\to x} F(t) = g(x)$ for $x \in X$,

[2] $F(x) = f(x)$ for $x \in X \setminus Y$.

C Let (X, ϱ) be a dense in itself metric space. For every upper semicontinuous function $f\colon X \to [0, \infty)$ there exists a function $F\colon X \to \mathbb{R}$ such that $\omega^*(F, x) = f(x)$ for $x \in X$.

For upper and lower Baire functions M_f and m_f theorem analogous to Theorem 2 is not true.

Example 2. Let $X = \{\frac{2k-1}{2^n} : k = 1 \ldots, 2^{n-1}, n \ge 0\} \subset \mathbb{R}$ and $f\colon X \to \mathbb{R}$, $f(\frac{2k-1}{2^n}) = 1 + \frac{1}{2^n}$ for $\frac{k}{2^n} \in X$. Then X is dense in itself, f is upper semicontinuous. Suppose, that there exists a function $F\colon X \to \mathbb{R}$ such that $M_F(x) = f(x)$ and $m_F(x) = 0$ for $x \in X$. Then $0 \le F \le f$. It is easy to prove that $\limsup_{t\to x} f(t) = 1$ for every $x \in X$. Hence $\limsup_{t\to x} F(t) \le 1$ for every $x \in X$. Since $M_F(x) = \max\{F(x), \limsup_{t\to x} F(x)\}$, it have to be $F(x) = f(x)$ for every $x \in X$. But then $m_F = 1$. Thus we have proved that there is **no** a function $F\colon X \to \mathbb{R}$ such that $M_F(x) = f(x)$ and $m_F(x) = 0$ for $x \in X$.

At the end we will consider problems of the existence of ω-primitives and ω^*-primitives for nonmetrizable topological spaces. The problem of the existence of ω-primitive has a positive solution for some nonmetrizable topological spaces, for example:

Theorem 10. Let $(X, \mathcal{T})$ be a regular separable topological space. Then for every upper semicontinuous function $f\colon X \to [0, +\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.

Theorem 11 ([?]). Let $(X, \mathcal{T})$ be a regular Baire space. Then for every upper semicontinuous function $f\colon X \to [0, +\infty]$ vanishing at isolated points of X there exists a function $F\colon X \to \mathbb{R}$ such that $\omega(F, \cdot) = f$.

The problem of the existence of ω^*-primitives for nonmetrizable topological spaces is more complicated.

Example 3. Let $(X, \mathcal{T})$, $X = \mathbb{R} \times [0, +\infty)$ be a Niemytzky plane. Then X is a "nice" nonmetrizabe, separable, Tychonoff, Baire topological space. Define $f\colon X \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & if \quad x \in \mathbb{Q} \times \{0\}, \\ 0 & if \quad x \notin \mathbb{Q} \times \{0\}. \end{cases}$$

We will show that ω^*-primitive for f does **not** exist. Let $F\colon X \to \mathbb{R}$ be any function such that $\omega^*(F, x) = f(x)$ for $x \in X \setminus (\mathbb{Q} \times \{0\})$. Then the function F has a limit at $(x, 0)$ for every $x \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$A_{n,k} = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : F(v) \in \left(\tfrac{k}{4} - \tfrac{1}{4}, \tfrac{k}{4} + \tfrac{1}{4}\right) \text{ for } v \in \left(x - \tfrac{1}{n}, x + \tfrac{1}{n}\right) \times \left(0, \tfrac{1}{n}\right) \right\}$$

for every $n, k \in \mathbb{N}$. Then $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n,k \in \mathbb{N}} A_{n,k}$ and by Baire Theorem there exist $n_0, k_0 \in \mathbb{N}$ and an open interval (a, b) such that A_{n_0,k_0} is dense in (a, b). But then for every $x_0 \in (a, b) \cap \mathbb{Q}$ there exists a neighbourhood U of $(x_0, 0) \in X$ such that $\sup_{u,v \in U \setminus \{(x_0,0)\}} |F(u) - F(v)| \le \frac{1}{2}$. Therefore $\omega^*(F, x_0) \le \frac{1}{2}$. Thus $\omega^*(F, x_0) \ne f(x_0) = 1$ and $\omega^*(F, \cdot) \ne f$. So, we have proved that ω^*-primitive for f does **not** exists.

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