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Students' Conception of a Point and its Relation to a Straght Line: A Comparison of Phylogenesis and Ontogenesis

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A point and a straight line are fundamental objects of Euclidean geometry which is taught at basic and secondary schools. Philosophers meditated on the nature of a point and a straight line long before Euclid (from the 6^{th} century BC). But it was Euclid (about 325 - 265 BC) who delimited the concept of a point and a straight line (and others) in the First book of his Elements (Stocheia) by means of a definition. The phylogenesis of a point and its relation to a straight line is marked out by names such as Viète, Kepler, Leibniz, Newton, Bolzano and Cantor.

Students meet the concept of a point before they start to create their geometrical structures. Our analysis will try to show that there is a strong parallel between the ontogenetic and phylogenetic aspects of the conception of a point and its relation to a straight line, a ray and/or a segment.

Historical Development of the Concepts

Constructional problems were the main concern of Greek geometry in 6^{th} and 5^{th} centuries BC. In this period we can find attempts to create geometrical structures and to formulate opinions about what is and what is not possible to construct. The concepts of a point and a straight line are specified. The focus of reasoning gradually moves from concrete problems towards ideal objects.

Pythagoras (580 - 500 BC) defines, via Proklos, a point as a monad that has a position. A point is thus a basic unit which has a placement. Pythagoras' school understands space as a sum total of points [Struik 1963, p. 42].

Parmenides (from Elea, about 530 - 470 BC) cogitates on how the world is recognizable. He compares the sensory and intellectual cognition. For example: What is a point? A dot on a paper? Or is the dot only a model of the point? A line drown on a paper isn't real geometrical object, it is only its model, its picture [Hejný 1985, pp. 50-51].

Zeno (from Elea, 490 - 430 BC) masterfully gives a true picture of the conflict between sensory perception and intellectual abstraction and idealization in his paradoxes.

Possibly as a response to Zeno's paradoxes, Democritus (470 - 360 BC) developed his mathematical atomism[1]. This theory posits an existence of an invisible unit which has a nonzero size and cannot be infinitely divided into ideal geometrical objects. All mathematical objects have direct material models; an ideal mathematical point and a straight line do not exist for Democritus [Drábek 1998, p.175].

Aristotle (from Stageira, 384 -322 BC) views and compares a unit and a point as monads, the former has no position and the latter has a position [Hejný 1985, p. 68].

Based on Plato's (427/8 - 347 BC) philosophy of ideas, Eudoxos (408 -390 BC) formulates his concept of continuous geometry, which culminates in Euclid's work. Euclid defines a point in the following way: ,,A point is that which has no part." (Semeion estin hu meros uthen.) He explains a point in accordance with Plato's conceptions of the world of ideas. So the idea of part is not present in the idea of a point. A placement only exists in the idea of a concrete point. Euclid also claims: "Borders of a line are points." (Grammes de perata semeia.) He defined a straight line[2] like ,,A line is a length without a wideness" and ,,A straight line is that which is straight to its points." Euclid does not highlight the continuity of a line (and other objects). Probably he considers it as evident. This explains his first postulate: ,,To lead a line from an arbitrary point to an arbitrary point." The second postulate says "To produce a finite straight line in a straight line." We see that Euclid understood a straight line ,,potentially infinity", continuous and ,,containing points".

Scientist of 16^{th} and 17^{th} century, like Kepler, Viète, Cavalieri or Torricelli, developed methods which leaded to infinitesimal calculus. They came back to Democritus' atomism. Cavalieri used theory of ,,indivisible elements" (in *Geometria indivisibilius continuorum, 1635*) that a straight line is formed by a movement of points [Struik 1963, p. 96].

Descartes discovered analytic geometry (in *Géométrie, 1637*), he used algebra in geometry. But abscissas were introduced by Leibnitz.

The set theory brought a new view on every geometrical object and became an ideal tool for investigating of phenomenon of infinity.

[1] Democritus' idea of a geometrical atom was used later in mathematical analysis by Viète and Kepler in the 16^{th} century.

[2] He used ,,eutheia" like a straight line, a ray and a segment too. If he needed to highlight a finite line he used ,,peperasmenès" and an infinity line he used ,,ep apeiron".

Phenomena of Phylogenesis of Concept of a Point

- a gradual movement from the pictorial form of representation of a point towards the abstract form
- a transition from a real object to an ideal one
- a determination of a position
- a size and a position
- a unit of a space
- smallness of a point
- isolation
- materiality of a point
- indivisibility of a point
- an endpoint of a segment
- an algebraic description (abscissas)

Methodology

Semi-structured interviews with 16 pupils and students from the age of 9 to 18 years were carried out in which the following kinds of questions were used:

1. What is a point?
2. Let us have the following problem: We have a segment AB. Divide it by means of a point C in the ratio of 2:3. Use scissors and cut the two parts which originated in the division by point C. What will the boundary points of both segments be?
3. How many points does consist a straight line (or a ray or a segment) of?

The first and third questions were used in a one-to-one interview (the experimenter spoke to one student only) and the second one was used in interviews in which two or three students spoke to the experimenter. The same words were used in follow-up questions as the subjects did earlier. The experimenter asked them about the size and the shape of a point and reasons for their answers. Among the follow-up questions was: What is the shortest distance between two different points? All the interviews were recorded and later transcribed.

Illustrations from the Interviews and our Comments

From the protocols of the interviews, the following extracts were chosen to illustrate some phenomena of the students' understanding of a point.

Interview No. 1: Girls M and L, boy J, 13 years old L has drawn a picture of a segment with two points.

E128: OK, what do we see? What does a point look like under a microscope?
L129: It is a small square, on the segment.
E130: Good. What is this, this, this and this one? (E shows some points on the square and its vertexes.)
M131: Points.
E132: Points?
L133: Yes, vertexes of the square. Of the small one.
E134: And is a vertex a point?
L135: Yes.

Comment: The students try to show how small a point is and to describe its shape. The point is a unit of the segment but it can have a „part".

Interview No 2: Boy J, 12 years old

E27: So, does a point have a size?
J28: It can have some size in reality. (He points to a mark on paper.)
E29: But I am not speaking about the drawn point. I am speaking about the proper geometrical point.
J30: (3 secs) So, the point's (5 secs) size is this. (He points to a mark on paper.)
E53: Do points B and D have a midpoint between themselves?
J54: Yes, they do.
E55: But you said that nothing was between them.
J56: Hmm. So the midpoint (3 secs) it has no size, so it can be there, in the nothing.

Comment: The boy feels that there are two different worlds - the physical and the geometrical world. An object which has zero size cannot exist in the physical world.

Interview No 3: One year later. Boy J, 13 years old

E1: What's a point?
J2: (6 secs) So, a point is (3 secs) a thing which determines some position, it demarcates something. (3 secs) It can't demarcate anything, it is somewhere.

J6: It doesn't have any size. It has no content. We can never draw a point, we can only indicate it.
J8: It's hard to imagine something that has no size.
E9: How do you know that a point has no size?
J10: ... If I pictured a point like a circle, I could divide it wherever I liked. And even if we then could not see it, it is still possible to divide it further.
J12: It is not possible to divide a point in this way. It demarcates only one place.

Comment: We can see a marked movement in the boy's understanding of geometrical objects. He can think and speak about ideal geometrical objects (and does not need a concrete point). He highlights a zero size, a position, an indivisibility and the demarcation of a point.

Phenomena of Ontogenesis of a Point

By analysing the protocols from the interviews, the following phenomena have been identified which characterize a student's understanding of a point:

- a gradual movement from the pictorial form of representation of a point towards the abstract form
- a transition from a real object to an ideal one
- a shape of a point
- smallness of a point
- isolation
- determination of a position
- a unit of a space
- indivisibility of a point
- an endpoint of a segment
- an algebraic description (abscissas)

Conclusions:

The Comparison of Phylogenesis and Ontogenesis

In general, students use similar concepts to those we can see in the historical development of the understanding of a point. There are two differences though. Students do not speak about a materiality of a point and, on the other hand, we have been unable to find any mention of a shape in its historical development. At the beginning, geometrical theories view a point as a shape. The emphasis laid on the materiality of a point in the phylogenesis corresponds to the priority attached to the tactile sense. Students tend to prefer the visual sense.

Many students are able to think about a point (and other geometrical objects) only by way of a picture. It corresponds to the first phase of the historical development of plane geometry, in which notions both of a real object and an ideal object are present. Pupils and students use them both and these ideas are sometimes contrasted with one another. Students become gradually able to „leave" the physical world and think in the ideal geometrical world.

All the pupils and students interviewed highlight the smallness of a point. But only some of them admit its zero size. It's a natural enough opinion as everything that we meet in everyday life has a finite size. Children do not understand a line or a segment as a "set" which is compactly ordered. This corresponds to Democritus' geometrical atomism. They emphasize that it is "only one place" like a unit of space.

Older (that is, in most cases, as it is an individual phenomenon) students speak about the determination of a position via a point. This can be partly due to school geometry that is taught, analytical geometry in particular.

Only a few students admit the indivisibility of a point. This is possible only if they stop thinking about a point as an object on paper.

Younger students (under 15 years old) often answered "*A straight line has no point*" or "*A ray has just one point.*" or "*A segment has just two points*". There are two reasons: They consider only about a picture: "*There are drawn two points.*" Or they can understand a "straight line with two points" like a segment and two rays. It corresponds to historical development of the relation between points and other geometrical objects. Points were understood like borders of a line or vertexes of a rectangle [Eisenmann 2002, pp. 9-10].

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