Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka IX, Częstochowa 2003

# The Justice of Chance as a Stochastic Dilemma

*Maciej Major, Barbara Nawolska*

**Abstract:** The paper deals with the problem of a fair draw of one or two elements out of $n$ elements. It presents an attempt to construct a generalisation of the term "a fair draw" (i.e. a draw in which it is equally likely to select each element) from the case of selection of one element of a certain population to the case where two elements are drawn.

In numerous situations we face the necessity of drawing one element from a finite set. The requirement is that in such an experiment the chances of being selected are equal for each element. A draw satisfying this condition will be called a fair draw. This type of situation occurs, for example, in the case of drawing a half of a playing field before a match or when a question in a competition is selected. It is put to practice with the aid of coins, matches, counting-out rhymes, ,,wheels of fortune", receptacles containing pieces of paper with names of people written on them etc.

In this context the following questions arise:

- How should a fair draw of one element from a given $n$-element set be organised?
- How should a fair draw of two or more elements from a given $n$-element set be organised?
- How could the fairness of such a draw be verified within the framework of mathematics?

Let us consider the following procedure of drawing.

**I.** From a group of $s$ people one has to be drawn in such a way that the chances of being selected are equal for each person. Quite often matches are used for this purpose. An organiser of the drawing holds matches in his or her hand; the head of one of the matches has been previously broken off. The matches are held in such a way that the participants of the experiment cannot see these parts of the matches where the head is or has been. Everyone from the group out of which one person has to be selected sequentially draw one match each. The person considered to be selected is the one who draws the match without a head.

Here the following question arises: Are the chances of being drawn equal for each participant? In other words, is such a draw, performed with matches, fair?

The drawing described above is concluded when the match without a head is drawn. It may seem that it is not fair, since the first person who draws a match always takes part in the drawing, whereas the person who is the last to draw has little chance of actually pulling out a match at all. Let us therefore alter the procedure slightly so that every person takes part in the drawing. We assume that, as previously, every person selects a match but without revealing the result. The person selected is indicated after the last match has been drawn. In both procedures the chances of being selected are equal for each person. The only factor the suggested modification influences is the duration of the whole procedure.

In order to justify the fairness of the draw a mathematical model of selection by drawing matches is constructed. We assume that $s = 4$.

Number 1 denotes the match without a head and number 0 a regular match. With this denotation the entire course of the modified experiment together with all its possible outcomes is described by the tree diagram in figure 1. Below each branch the result of the draw is noted.

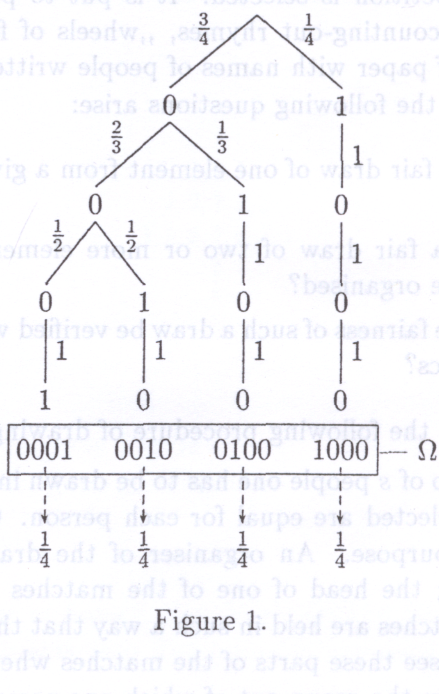


Figure 1.

By means of the tree diagram a classical sample space is constructed. The suggestion that it is the model of the considered experiment is supported by the observation that, in fact, we draw one of the four places for the match without a head and selecting each of these places is equally probable. Every outcome of the draw uniquely determines the selected person. All outcomes are equally likely, therefore the chances of being selected are equal for each person.

The reasoning presented above may be easily generalised to the case of $s$ persons.

In the case of $s$ persons the selection by drawing matches may be replaced by drawing a ball without replacement from a box containing $(s-1)$ white balls and one black ball. Such a draw may be put to practice in two ways:

— all participants sequentially draw one ball from the box and the person who pulls out the black ball is considered to be selected,

— the participants are numbered, from the box a ball is drawn $s$ times without replacement and the person selected is the one whose number is the same as the number of the step at which the black ball has been drawn.

**II**. Let us now consider the following procedures of drawing a representation of two from a group of four persons.

1. We take four matches, among which there are two without a head. The persons from which two have to be selected sequentially draw one match each until both matches without a head have been selected. Those who drew matches without a head are the ones who are selected.

2. We take four matches, among which one has no head. First, one person from four is selected. Then we take three matches - two regular ones and one without a head - and out of the remaining three persons we select one with the aid of the matches.

3. One of the four matches has no head. Before the drawing begins all participants stand in a circle. Then, starting from a previously assigned person, they draw matches sequentially (for example, clockwise). The participants selected are the person who drew the match without a head and the next person (according to the order of drawing).

Let us consider the following options:

— the participants in a circle are ordered alphabetically (with regard to their surnames),

— the order of the participants was determined as a result of a fair draw.

Let us assume that four persons A, B, C and D take part in the experiment and that they draw matches in the alphabetical order.

Now we consider the first method of drawing.

The course of the experiment and all its possible outcomes is represented by the tree diagram in figure 2. Similarly to the previous diagram, below each branch the result of the draw is noted.

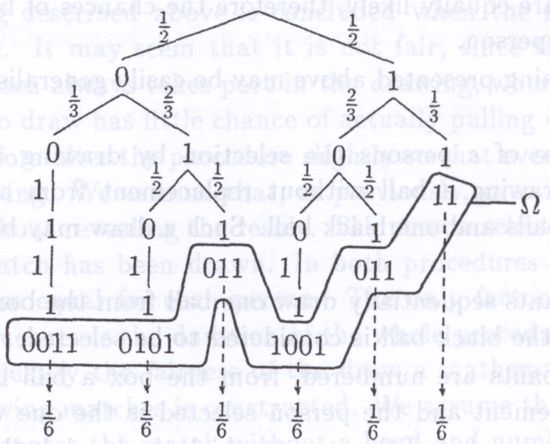

Figure 2.

By means of the stochastic tree diagram a classical sample space is constructed. In this case the results are sequences of various lengths (with two, three or four terms), in which exactly two terms equal 1. If this procedure were modified in the same way as in the previous case, that is, so that all participants take part in the drawing without revealing who drew a match without a head, then all the results would be fourterm sequences in which exactly two terms equal 1. In this situation the actual problem lies in selecting two places out of four for matches without a head (i.e. for „1"). Similarly to the case of drawing one person out of four, the suggested modification influences neither the result of the draw nor either of the participants' chances of being selected; it may only affect the duration of the experiment.

The probability distribution on the set $\Omega$ becomes the following function $p$:

| $\omega \in \Omega$ | 11 | 011 | 101 | 0011 | 0101 | 1001 |
|---|---|---|---|---|---|---|
| $p(\omega)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

The following events are associated with the experiment:

$A = \{person\ A\ was\ drawn\}$,

$B = \{person\ B\ was\ drawn\}$,

$C = \{person\ C\ was\ drawn\}$,

$D = \{person\ D\ was\ drawn\}$,

$E_{A-B} = \{person\ A$ and $person\ B\ were\ drawn\}$,

$E_{A-C} = \{person\ A$ and $person\ C\ were\ drawn\}$,

$E_{A-D} = \{person\ A$ and $person\ D\ were\ drawn\}$,

$E_{B-C} = \{person\ B$ and $person\ C\ were\ drawn\}$,

$E_{B-D} = \{person\ B$ and $person\ D\ were\ drawn\}$,

$E_{C-D} = \{person\ C$ and $person\ D\ were\ drawn\}$.

In the sample space $(\Omega, p)$ we obtain:

$$A = \{11, 101, 1001\}, \quad B = \{11, 011, 0101\},$$
$$C = \{011, 101, 0011\}, \quad D = \{0011, 0101, 1001\},$$
$$E_{A-B} = \{11\}, \quad E_{A-C} = \{101\}, \quad E_{A-D} = \{1001\},$$
$$E_{B-C} = \{011\}, \quad E_{B-D} = \{0101\}, \quad E_{C-D} = \{0011\}.$$

Let us notice that

$$P(A) = P(B) = P(C) = P(D) = \frac{1}{2}$$

and

$$P(E_{A-B}) = P(E_{A-C}) = P(E_{A-D}) = P(E_{B-C}) =$$
$$= P(E_{B-D}) = P(E_{C-D}) = \frac{1}{6}.$$

For this reason, the chances of being drawn are equal for each person. Moreover, the chances of selecting each pair of persons are the same. The draw may therefore be considered to be fair.

Let us now consider the second method of drawing. The fact that selection of one person out of $s$ by drawing matches is a fair draw will be employed.

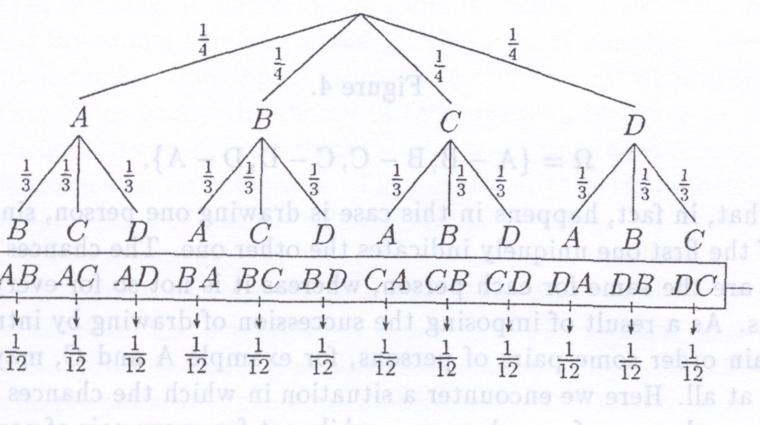


Figure 3.

$$\Omega = \{AB, AC, AD, BA, BC, BD, CA, CD, CB, DA, DB, DC\}.$$

The outcomes of this experiment are two-term sequences of different elements of the set $\{A, B, C, D\}$. With the aid of the stochastic tree diagram a classical sample space is constructed. Certain symmetries support the assumption that each outcome of this experiment is equally probable. For this reason, the sample space constructed in the above way may be regarded as a model of the discussed experiment.

Each outcome of the draw uniquely determines the person selected. All the outcomes are equally likely, so the chances of being selected are equal for each person.

The symmetry of the tree diagram implies that in this situation every person's chances are the same and the chances of selecting each pair of persons are the same. It means that the two presented methods of drawing may be used interchangeably.

Let us, finally, consider the third method of drawing.

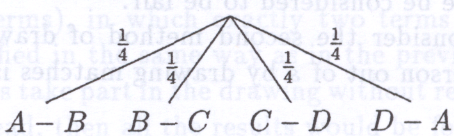


Figure 4.

$$\Omega = \{A - B, B - C, C - D, D - A\}.$$

What, in fact, happens in this case is drawing one person, since selection of the first one uniquely indicates the other one. The chances of being drawn are the same for each person, whereas it is not so for every pair of persons. As a result of imposing the succession of drawing by introducing a certain order some pairs of persons, for example A and C, may not be drawn at all. Here we encounter a situation in which the chances of being drawn are the same for each person, while not for every pair of persons are these chances equal.

The third procedure may not be used interchangeably with either of the two previous ones. Therefore, let us modify this procedure by exchanging the assumption: *the persons have been ordered according to some rule*, for the assumption: *the order of the participants is the result of a fair draw.* It may be easily verified that under the new assumption all three presented methods may be applied interchangeably.

## References

[1] M. Major, B. Nawolska, *Matematyzacja, dedukcja, rachunki i interpretacja w zadaniach stochastycznych*, Wydawnictwo Naukowe WSP, Kraków 1999.

[2] A. Płocki, *Stochastyka 1. Rachunek prawdopodobieństwa i statystyka matematyczna ,,in statu nascendi"*, Wydawnictwo Naukowe WSP, Kraków 1997.

Institute of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Kraków
Poland
e-mail: mmajor@ap.krakow.pl

Department of Preschool and School Education
Pedagogical University of Cracow
Ingardena 4
30-060 Kraków
Poland
e-mail: bnawol@ap.krakow.pl