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ON ASSOCIATIVE RATIONAL FUNCTIONS WITH ADDITIVE GENERATORS

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Abstract

We consider the class of rational functions defined by the formula

$$F(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where φ is a homographic function and we describe all associative functions of the above form.

1. Introduction

The functional equation of the form

$$f(x+y) = F(f(x), f(y)),$$

where F is an associative rational function is called an addition formula. For the rational two-place real-valued function F given by

$$F(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)),$$

where φ is a homographic function (such F is called a function with an additive generator), the addition formula has the form

$$\varphi(f(x+y)) = \varphi(f(x)) + \varphi(f(y))$$

and it is a conditional functional equation if the domain of φ is not equal to $\mathbb{R}$. Solutions of the above conditional equation are functions of the homographic type. Some results on such equations can be found in the article [2]. It seems to be interesting which homographic functions lead F of above form to be associative.

The following lemma will be useful in the sequel.

Lemma. *Let $A, B, C, D \in \mathbb{R}$ be given and let $AD \neq BC$, $C \neq 0$. For φ given by*

$$\varphi(x) = \frac{Ax+B}{Cx+D},$$

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holds

$$\varphi^{-1}(\varphi(x)+\varphi(y)) = \frac{C(BC-2AD)xy - AD^2(x+y) - BD^2}{AC^2xy + BC^2(x+y) + (2BC-AD)D}.$$

Proof. We have

$$\varphi(x)+\varphi(y) = \frac{2ACxy + (AD+BC)(x+y) + 2BD}{C^2xy + CD(x+y) + D^2}$$

and

$$\varphi^{-1}(x) = \frac{-Dx+B}{Cx-A}.$$

A simple calculation shows that the above equation holds true. □

2. Main results

We proceed with a description of the class of rational functions with additive generators.

Theorem 1. *The following functions (with natural domains in question) are the only associative members of the class of rational functions with additive generators:*

$$F(x,y) = \frac{xy}{\alpha xy + x + y} \quad , \quad \alpha \in \mathbb{R};$$

$$F(x,y) = \frac{x+y+2\lambda xy}{1-\lambda^2 xy} \quad , \quad \lambda \neq 0;$$

$$F(x,y) = \frac{(1-2\alpha\beta)xy - \alpha\beta^2(x+y) - \beta^2}{\alpha xy + x + y + 2\beta - \alpha\beta^2} \quad , \quad \alpha \in \mathbb{R},\ \beta \neq 0.$$

Proof. Assume that F is associative and has the additive generator

$$\varphi(x) = \frac{Ax+B}{Cx+D},$$

where $A, B, C, D \in \mathbb{R}$ and $AD \neq BC$, $C \neq 0$.

We infer from Lemma that

$$F(x,y) = \frac{C(BC-2AD)xy - AD^2(x+y) - BD^2}{AC^2xy + BC^2(x+y) + (2BC-AD)D} \tag{1}$$

First assume that $D = 0$. If $B = 0$ then $AD = BC$ in contradiction to the assumption. Hence $B \neq 0$. Putting $D = 0$ in (1) we obtain

$$F(x,y) = \frac{BC^2xy}{AC^2xy + BC^2(x+y)}$$

Consequently putting

$$\alpha = \frac{A}{B},$$

we infer that

$$F(x,y)=\frac{xy}{\alpha xy+x+y},\qquad \alpha\in\mathbb{R}.$$

Let now $D\neq 0$. We get

$$\varphi(x)=\frac{\hat{A}x+\hat{B}}{\hat{C}x+1},$$

where

$$\hat{A}=\frac{A}{D},\hat{B}=\frac{B}{D},\hat{C}=\frac{C}{D}.$$

Replacing $\hat{A}$ by A, $\hat{B}$ by B and $\hat{C}$ by C we have by (1)

$$F(x,y)=\frac{C(BC-2A)xy-A(x+y)-B}{AC^2xy+BC^2(x+y)+2BC-A}. \tag{2}$$

From Theorem 1. (proved in article [1]) we obtain that every F of the above form is associative. Consequently, in the case $B=0$ (if $B=0$ then $A\neq 0$) we have

$$F(x,y)=\frac{-2ACxy-A(x+y)}{AC^2xy-A}=\frac{x+y+2\lambda xy}{1-\lambda^2xy}$$

with $\lambda=C$.

In the case $B,C\neq 0$ in (2) we have

$$F(x,y)=\frac{\left(1-2\frac{A}{B}\cdot\frac{1}{C}\right)xy-\frac{A}{B}\cdot\frac{1}{C^2}(x+y)-\frac{1}{C^2}}{\frac{A}{B}xy+x+y+2\frac{1}{C}-\frac{A}{B}\cdot\frac{1}{C^2}}=$$

$$=\frac{(1-2\alpha\beta)xy-\alpha\beta^2(x+y)-\beta^2}{\alpha xy+x+y+2\beta-\alpha\beta^2}$$

with $\alpha=\frac{A}{B}$, $\beta=\frac{1}{C}$.

It is easy to check (see Theorem 2 or Theorem 1 in [1]) that each of the function above yields a rational associative function. Thus the proof has been completed. □

Now we indicate homographic functions φ which by the formula

$$F(x,y)=\varphi^{-1}(\varphi(x)+\varphi(y))$$

lead to associative functions F.

Theorem 2. *For following homographic functions (with natural domains in question) we obtain all rational associative functions with an additive generator*

$$\varphi(x)=\frac{1}{x}$$

$$\varphi(x)=\frac{\alpha x+1}{\alpha x}$$

$$\varphi(x) = \frac{\lambda}{x+\lambda}$$
$$\varphi(x) = \frac{\lambda x}{\lambda x+1}$$
$$\varphi(x) = \beta\frac{\alpha x+1}{x+\beta},$$

where $\alpha, \beta, \lambda \in \mathbb{R} \setminus \{0\}$ *are arbitrary constants.*

Proof. It is easy to check that each of the function above yields a generator to the rational function which is associative. Moreover they are generators of

$$F(x,y) = \frac{xy}{x+y};$$
$$F(x,y) = \frac{xy}{\alpha xy + x + y} \quad , \quad \alpha \neq 0;$$
$$F(x,y) = \frac{xy - \lambda^2}{x + y + 2\lambda} \quad , \quad \lambda \neq 0;$$
$$F(x,y) = \frac{x + y + 2\lambda xy}{1 - \lambda^2 xy} \quad , \quad \lambda \neq 0;$$
$$F(x,y) = \frac{(1-2\alpha\beta)xy - \alpha\beta^2(x+y) - \beta^2}{\alpha xy + x + y + 2\beta - \alpha\beta^2} \quad , \quad \alpha, \beta \neq 0.$$

respectively. From Theorem 1 it is the assertion of Theorem 2. Thus the proof has been completed. □

References

[1] K. Domańska, *An analytic description of the class of rational associative functions*, Annales Universitatis Paedagogicae Cracoviesis Studia Mathematica 11 (2012), 111-122.

[2] K. Domańska, *On some addition formulas for homographic type functions*, Scientific Issues, Jan Długosz University in Częstochowa, Mathematics XVII, (2012) 17-24.

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