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The No Arbitrage Conditions in the Segmented Markets

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The present paper is the direct continuation of papers Medvedev (2001, 2002). In the financial market the term arbitrage refers to the possibility of making a trading gain with no chance of loss. The idea expressed by the no arbitrage condition consists that in the equilibrium market two portfolios of securities, which ensure identical payments, should have in each instant the identical price. Intuitively it is clear that such a definition of the price excludes the arbitrage. The arbitrage theory of market asset pricing is recently popular. It bases on the assumption that the financial market is arbitrage-free. To check the fulfillment of such assumption it is necessary to have some "no arbitrage" conditions. This explains the interest to derive such conditions.

In the first paper it was set no arbitrage condition for the market with any number risky assets and single random market factor. In the second paper this problem was considered when the inflation is also in a market. Now this problem is solved for a case when the market is divided on some segments according to the maturity terms.

In the present paper the no arbitrage conditions are derived in the explicit form for the market, where the zero coupons bonds of n various maturities are accessible for the investors to draw up the portfolios. It is supposed that the investor at any moment of time t has a possibility to make the self-financed portfolio of value $S(t)$ by including in this portfolio the bonds with maturities T_j, $1 \leq j \leq n$, on the sum S_j. $S(t) = \sum_{j=1}^{n} S_j$. It is considered that the processes of the short interest rate and rates of inflation follow the stochastic differential equations, and the prices of the bonds are expressed by functions that have the mathematical derivatives of the necessary orders.

The arguments of the previous papers are agreed with the theory of the term structure of the interest rates based on a so-called expectations hypothesis. According to this theory, the forward interest rates are considered as unbiased estimates of expected future short interest rates. Therefore, it is natural to suppose that in the equations for processes of the bond prices with any maturity it is possible to use the same equation of process of the short interest rate. However, not always the results of this theory will agree with market realities. In this connection there are also other theories of term structure (Hull, 1993). According to the theory of market segmentation there are simultaneously some independent processes of the short interest rates appropriate to various maturity terms and controlled by the supply and demand for assets with these terms. More often assets, which are traded in the market, are divided into three segments: short-term, intermediate term and long-term assets. Due to independence of mechanisms of installation of the prices on assets into each from the segments, it is possible to assume that the independent Wiener processes generate the equilibrium processes of the short-term interest rates inside of various segments. Consider no arbitrage conditions in this case.

Still we shall assume that in the financial market there are traded the zero coupon bonds with maturity dates that form a set $\{T_j, 1 \leq j \leq n\}$. Let's assume also, that this maturity set is divided into m of nonintersecting segments $\mathbf{T}_k$, $1 \leq k \leq m$, $m < n$. Inside the k-th segment the actual short interest rate $r_k(t)$ follows the stochastic differential equation

$$dr_k(t) = \mu_{kr}(r(t), t)dt + \sigma_{kr}(r(t), t)dW_k(t), \quad 1 \leq k \leq m.$$

According to this the yield of the bond with maturity date $T_j \in \mathbf{T}_k$ is determined by the stochastic differential equation

$$\frac{dP(t,T)}{P(t,T)} = \mu^{(j)}(r_k, t)dt + \sigma^{(j)}(r_k, t)dW_k(t). \tag{1}$$

Under the inflation instead of the actual interest rate $r_k(t)$ in the equation (1) it is necessary to use the nominal interest rate $R_k(t) = (1 + r_k(t))(1 + i(t)) - 1$, where the rate of inflation varies according to process

$$di(t) = \mu_i(i(t), t)dt + \sigma_i(i(t), t)dW_i(t).$$

As the mechanisms underlying a stochastic change of the processes $r(t)$ and $i(t)$ are generally different and in the certain degree are independent, the processes $W_r(t)$ and $W_i(t)$ are also different and can be dependent only somewhat.

In this case equation (1) should be modified to a form, taking into account a stochastic behavior of the rate of inflation:

$$\frac{dP(t,T)}{P(t,T)} = \mu^T(t)dt + \sigma_r^T(t)W_1(t) + \sigma_i^T(t)W_2(t),$$

where the arguments r and i at functions μ and σ are omitted for brevity. These functions are determined by the formulae

$$\sigma_r^T(t) = \sigma_r(r,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial r},$$

$$\sigma_i^T(t) = \sigma_i(i,t)\frac{1}{P(r,i,t,T)}\frac{\partial P}{\partial i},$$

$$\mu^T(t) = \frac{1}{P}\Big(\frac{\partial P}{\partial t} + \mu_r\frac{\partial P}{\partial r} + \mu_i\frac{\partial P}{\partial i} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_i^2\frac{\partial^2 P}{\partial i^2}\Big).$$

However, as the purpose of the present paper is to find the no arbitrage conditions for the segmented market, we shall not take inflation into account.

As well as until now we assume that the investor makes a portfolio of the bonds of value $S(t)$ purchasing N_j bonds with maturity term T_j on the price $P(t,T_j)$, so that $S_j = N_jP(t,T_j)$,

$$S(t) = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} N_jP(t,T_j).$$

Using method, which was used in the previous papers, it is possible to obtain an increment of this portfolio value of the bonds for an infinitesimal time interval as

$$dS(t) = \sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}\mu^{(j)}(r_k,t)S_j dt + \sum_{k=1}^{m}\sum_{j=1}^{n} I_{kj}\sigma^{(j)}(r_k,t)S_j dW_k(t), \qquad (2)$$

where I_{kj} is the indicator

$$I_{kj} = \begin{cases} 1, & \text{if } \ T_j \in \mathbf{T}_k \\ 0, & \text{if } \ T_j \in \mathbf{T}_k \end{cases}$$

Again, for deriving the risk-free profit it is necessary that $\{S_j\}$ are selected so that the stochastic terms in (2) are equal to zero. Therefore, we have m conditions of risk-free deriving of interests

$$\sum_{j=1}^{n}[I_{kj}\sigma^{(j)}(r_k,t)]S_j = 0, \quad 1 \le k \le m. \qquad (3)$$

The equalities (3) are the existence conditions of risk-free self-financed portfolio. As well as in the previous section, in order to obtain the no arbitrage condition, it is necessary to add a demand that the self-financed portfolio can earn inside each of m market segments only at the risk-free interest rate $r_k(t), 1 \leq k \leq m$, i.e.

$$\sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}\mu^{(j)}(r_k,t)S_j = \sum_{k=1}^{m}\sum_{j=1}^{n} I_{kj}r_k(t)S_j,$$

hence

$$\sum_{j=1}^{n}\sum_{k=1}^{m} I_{kj}[\mu^{(j)}(r_k,t) - r_k(t)]S_j = 0. \qquad (4)$$

Thus, the no arbitrage condition is held if for any set $\{Sj\}$ the equalities (3) and (4) are simultaneously fulfilled. In vector-matrix notation this means

$$\begin{bmatrix} \sum_{k=1}^{m} I_{k1}[\mu^{(1)}(r_k,t) - r_k(t)] & \dots & \sum_{k=1}^{m} I_{kn}[\mu^{(n)}(r_k,t) - r_k(t)] \\ \sigma^{(1)}(r_1,t) & \dots & \sigma^{(n)}(r_1,t) \\ \dots & \dots & \dots \\ \sigma^{(1)}(r_m,t) & \dots & \sigma^{(n)}(r_m,t) \end{bmatrix} \times$$

$$\times \begin{bmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Because this equality must be fulfilled for any $\{Sj\}$, therefore the determinant of the matrix must be equal to zero. Then the rows of matrix are linear dependent, i.e.

$$\sum_{k=1}^{m}[\mu^{(j)}(r_k,t) - r_k(t)]I_{kj} = \sum_{k=1}^{m}\lambda_k(r_k,t)\sigma^{(j)}(r_k,t)I_{kj}, \quad 1 \leq j \leq n. \qquad (5)$$

As the bond with some specific maturity date T_j can belong only to one segment, the sums in equality (5) contain only a single nonzero term for this maturity. Let us determine a set J of pairs (j,k) for which $I_{kj} = 1$, i.e. $(j,k) \in J$ if a security with maturity date T_j is traded at the k-th segment of financial market. Then we can formulate the no arbitrage condition in the form of the following proposition.

The no arbitrage conditions are held in the segmented market if for each $(j,k) \in J$ the following equality is fulfilled

$$\frac{\mu^{(j)}(r_k,t) - r_k(t)}{\sigma^{(j)}(r_k,t)} = \lambda_k(r_k,t).$$

It means, that for each segment of the financial market there is a market price of risk, identical for the bonds of all maturity dates of this segment, but in general various for various segments.

References

[1] J. C. Hull, *Options, Futures, and other Derivative Securities.* Englewoods Cliffs: Prentice Hall, 1993.

[2] G. A. Medvedev, *No arbitrage conditions for markets with arbitrary number of securities,* Acta Universitatis Purkynianae. Studia mathematica. Usti n/L, V. 72, pp. 26-31, 2001.

[3] G. A. Medvedev, *No arbitrage condition for the financial market with inflation.* 9-th Czech - Polish Mathematical School. Krakow, 2002.

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