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On Some Stability Properties of Monomial Functions

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Abstract

A map M defined on a semigroup (group, Banach space etc.) and taking values in an Abelian group is called monomial of degree at most n whenever

$$\triangle_y^n M(x) = n!M(y).$$

We deal with the following stability problem for monomial mappings: given two functions F and f satisfying the inequality

$$\| n!F(y) - \triangle_y^n F(x) \| \leq n!f(y) - \triangle_y^n f(x),$$

we are looking for conditions admitting the existence of a nonnegative constant α such that

$$\| F(x) \| \leq f(x) + \alpha \cdot \| x \|^n .$$

1. Introduction

Given functions F and f satysfying inequality

$$\| F(x+y) - F(x) - F(y) \| \leq f(x) + f(y) - f(x+y)$$

(resp.

$$\| F(x+y) + F(x-y) - 2F(x) - 2F(y) \| \leq 2f(x) + 2f(y) - f(x+y) - f(x-y))$$

R. Ger was looking in [3] for conditions implying the existence of a constant c such that

$$\|F(x)\| \leq f(x) + c \cdot \| x \|$$

$$(\text{resp.} \quad \| F(x) \| \leq f(x) + c \cdot \| x \|^2).$$

We consider, for a function f mapping a semigroup $(S,+)$ into an Abelian group $(X,+)$ and for a fixed $y \in S$, the well-known difference operator $\triangle_y$ which is defined recurrently by

$$\triangle_y^1 f(x) = f(x+y) - f(x)$$

and, for a positive integer n, by

$$\triangle_y^{n+1} f(x) = \triangle_y^1 \triangle_y^n f(x),$$

where $x \in S$.

It is well-known (see e.g. L. Székelyhidi [5] or M. Kuczma [4]) that the explicite form of the n-th iterate of $\triangle$ reads as follows:

$$\triangle_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

Moreower, for any two Abelian groups $(X,+)$ and $(Y,+)$ one has

Lemma. *Let $F : X^k \longrightarrow Y$ be a symmetric k-additive function and let $f : X \longrightarrow Y$ be the diagonalization of F, i.e.*

$$f(x) := F(x, \ldots, x), \; x \in X.$$

Then for every $n \in \mathbb{N}$, $n \geq k$, and for every $x, h \in X$ we have

$$\triangle_h^n f(x) = \begin{cases} k! F(h, \ldots, h) & \text{if } n = k \\ 0 & \text{if } n > k. \end{cases}$$

A map $f : S \longrightarrow X$ is called a *monomial of degree at most n* if and only if

$$\triangle_y^n f(x) - n! f(y) = 0$$

for all $x, y \in S$.

Observe that additive mappings and quadratic mappings are monomial functions of degree at most 1 and 2, respectively.

In [1] we have proved the following

Theorem 1. *Let $(X, \|\cdot\|)$ be a real Banach space and $(Y, \|\cdot\|)$ be a real normed linear space. Suppose that $f : X \longrightarrow \mathbb{R}$ is a continuous functional such that $\triangle_y^n f(x) \geq 0$ for all $x, y \in X$ and $F : X \longrightarrow Y$ is a continuous mapping such that inequality*

$$\| n! F(y) - \triangle_y^n F(x) \| \leq n! f(y) - \triangle_y^n f(x) \tag{1}$$

holds true for all $x, y \in X$. Then there exists a nonnegative constant c such that

$$\| F(x) \| \leq c \cdot \| x \|^n + f(x), \; x \in X.$$

In the present paper we continue the study of the functional inequality

$$\| n!F(y) - \Delta_y^n F(x) \| \leq n!f(y) - \Delta_y^n f(x), \tag{2}$$

looking for new conditions admitting the existence of a nonnegative constant α such that

$$\| F(x) \| \leq f(x) + \alpha \cdot \| x \|^n .$$

2. Stability properties of monomial functions

Let X, Y be two real normed linear spaces. With the aid of the Lemma it is easily seen that for every $k-$additive and symmetric mapping $M : X^k \longrightarrow Y$, the diagonalization $m : X \longrightarrow Y$ given by the formula

$$m(x) = M(x, \ldots, x), \; x \in X,$$

yields a monomial of degree at most k (see also M. Kuczma [4]).

In what follows, $D^k g(x)$ will stand for the $k-$th Fréchet differential of a map g at a point x; plainly, $D^k g(x)$ is a $k-$additive and symmetric mapping. The monomial generated by $D^k g(x)$ will be denoted by $d^k g(x)$.

We shall use the following version of Taylor's formula (see e.g. J. Dieudonné [2]).

Theorem 2. *Let $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ be two real Banach spaces. Let further $F : X \longrightarrow Y$ be an $n-$times continuosly differentiable function and let $x_0 \in X$. Then, for every $x \in X$, we have*

$$F(x) = \sum_{k=0}^{n-1} \frac{1}{k!} d^k F(x_0)(x - x_0) + R(x),$$

where

$$R(x) = \int_0^1 \frac{(1-\xi)^{n-1}}{(n-1)!} d^n F(x_0 + \xi(x - x_0))(x - x_0) d\xi.$$

Moreover, if there exists a constant c such that

$$\| d^n F(x) \| \leq c, \; x \in X,$$

then

$$\| R(x) \| \leq \frac{c}{n!} \| x - x_0 \|^n, \; x \in X.$$

Now, we are in a position to state our main result.

THEOREM. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real Banach spaces. Let further $F: X \longrightarrow Y$ and $f: X \longrightarrow \mathbb{R}$ be two C^n-mappings such that inequality*

$$\| n!F(y) - \Delta_y^n F(x) \| \leq n!f(y) - \Delta_y^n f(x) \tag{3}$$

holds true for all $x, y \in X$. If there exist constants c and C such that

$$\| d^n F(x) \| \leq C, \quad x \in X, \tag{4}$$

and

$$\| d^n f(x) \| \leq c, \quad x \in X, \tag{5}$$

then there exists a nonnegative constant α such that

$$\| F(x) \| \leq \alpha \cdot \| x \|^n + f(x), \quad x \in X.$$

P r o o f. Assume that $F: X \longrightarrow Y$ and $f: X \longrightarrow \mathbb{R}$ satisfy (3). By virtue of Theorem 2 applied for $x_0 = 0$ we obtain

$$F(x) = \sum_{k=0}^{n-1} \frac{1}{k!} d^k F(0)(x) + R(x), \quad x \in X,$$

with

$$\| R(x) \| \leq \frac{1}{n!} C \cdot \| x \|^n, \quad x \in X, \tag{6}$$

(see (4)).

Applying Theorem 2 again we infer that, for every $x \in X$, one has

$$f(x) = \sum_{k=0}^{n-1} \frac{1}{k!} d^k f(0)(x) + r(x),$$

where

$$r(x) = \int_0^1 \frac{(1-\xi)^{n-1}}{(n-1)!} d^n f(\xi x)(x) d\xi.$$

Hence, by (5) we obtain

$$|r(x)| \leq \frac{1}{n!} c \cdot \| x \|^n, \quad x \in X. \tag{7}$$

By virtue of the Lemma, we infer that

$$\triangle_y^n F(x) = \triangle_y^n R(x), \quad x, y \in X \tag{8}$$

and

$$\triangle_y^n f(x) = \triangle_y^n r(x), \quad x, y \in X. \tag{9}$$

On account of the explicite form of $\triangle^n$ (see the Introduction) we infer that

$$\| \triangle_y^n R(x) \| \leq \sum_{k=0}^{n} \binom{n}{k} \| R(x+ky) \|, \quad x, y \in X, \tag{10}$$

as well as

$$| \triangle_y^n r(x) | \leq \sum_{k=0}^{n} \binom{n}{k} |r(x+ky)|, \quad x, y \in X. \tag{11}$$

From (3) we deduce that

$$\| n!F(y) \| - \| \triangle_y^n F(x) \| \leq n!f(y) - \triangle_y^n f(x), \quad x, y \in X,$$

whence

$$\| n!F(y) \| \leq n!f(y) + \| \triangle_y^n F(x) \| - \triangle_y^n f(x), \quad x, y \in X. \tag{12}$$

Fix arbitrarily $x, y \in X$. Then, by (8)-(12), one obtains

$$\begin{aligned} \| n!F(y) \| &\leq n!f(y) + \| \triangle_y^n F(x) \| - \triangle_y^n f(x) \leq \\ &\leq n!f(y) + \| \triangle_y^n F(x) \| + |\triangle_y^n f(x)| = \\ &= n!f(y) + \| \triangle_y^n R(x) \| + |\triangle_y^n r(x)| \leq \\ &\leq n!f(y) + \sum_{k=0}^{n} \binom{n}{k} \| R(x+ky) \| + \\ &+ \sum_{k=0}^{n} \binom{n}{k} |r(x+ky)|. \end{aligned}$$

In particular, taking here $x = y$ we get

$$\| n!F(x) \| \leq n!f(x) + \sum_{k=0}^{n} \binom{n}{k} \| R((k+1)x) \| + \sum_{k=0}^{n} \binom{n}{k} |r((k+1)x)|.$$

Hence, by (6) and (7) we have

$$\begin{aligned} \| n!F(x) \| &\leq n!f(x) + \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n!} [C \| (k+1)x \|^n + c \| (k+1)x \|^n] \\ &= n!f(x) + \sum_{k=0}^{n} \binom{n}{k} \frac{C+c}{n!} (k+1)^n \cdot \| x \|^n, \quad x \in X, \end{aligned}$$

which implies that

$$\| F(x) \| \leq f(x) + \sum_{k=0}^{n} \frac{(k+1)^n (C+c)}{k!(n-k)!n!} \cdot \| x \|^n, \quad x \in X.$$

Putting

$$\alpha := \frac{(C+c)}{n!} \sum_{k=0}^{n} \frac{(k+1)^n}{k!(n-k)!},$$

we get

$$\| F(x) \| \leq f(x) + \alpha \cdot \| x \|^n, \quad x \in X,$$

which completes the proof.

Remark 1. It easy to see that, for instance, the function

$$f(x) = m(x) + (a(x))^{2k}, \quad x \in X,$$

where $a : X \longrightarrow \mathbb{R}$ is linear, $2k < n$, $n, k \in \mathbb{N}$, $m : X \longrightarrow \mathbb{R}$ is a continuous monomial function of degree at most n, satisfies the inequality $n!f(y) - \Delta_y^n f(x) \geq 0$ for all $x, y \in X$ and $d^n f$ is constant (hence bounded) but not necessarily $\Delta_y^n f(x) \geq 0$ for all $x, y \in X$ (see Theorem 1 above).

In particular, for even n, the function

$$f(x) = \alpha x^n + \beta x^{2k}, \quad x \in \mathbb{R}, \ \alpha < 0, \ \beta > 0, \ 2k < n,$$

and for odd n, the function

$$f(x) = \alpha x^n + \beta x^{2k}, \quad x \in \mathbb{R}, \ \alpha \neq 0, \ \beta > 0, \ 2k < n,$$

satisfy the inequalities $n!f(y) - \Delta_y^n f(x) \geq 0$ for all $x, y \in \mathbb{R}$ and $\| d^n f(x) \| \leq c$ for any $c \in \mathbb{R}$ and all $x \in \mathbb{R}$, but the inequality $\Delta_y^n f(x) \geq 0$ fails to hold for all $x, y \in \mathbb{R}$.

Remark 2. If one drops the assumption $\| d^n F(x) \| \leq C$, $x \in X$ and $\| d^n f(x) \| \leq c$, $x \in X$, then the theorem fails to hold. In fact, take $X = Y = \mathbb{R}$, $n = 2$, $F(x) = 0$, $x \in \mathbb{R}$, and $f(x) = -x^4$, $x \in \mathbb{R}$. Then, for all $x, y \in \mathbb{R}$, one has

$$0 = \| 2!F(y) - \Delta_y^2 F(x) \| \leq 12y^2(x+y)^2 = 2!f(y) - \Delta_y^2 f(x),$$

but there exists no nonnegative constant α such that

$$0 = |F(x)| \leq \alpha \cdot x^2 - x^4, \ x \in \mathbb{R}.$$

Observe that there exists no nonnegative constant c such that $\| d^2 f(x) \| \leq c$, $x \in \mathbb{R}$.

References

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