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The Notion of Cube in the Theory of Squares*

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In this paper we define the notion of cube which is the primitive term of cube geometry of E. Glibowski and J. Słupecki (see [5]). In this way we use only the terms: "$\leq$", "K" and "$\equiv$" as primitive. The expressions: $X \leq Y$, K(X) and $X \equiv Y$ are read: the object X is a proper or improper part of the object Y, the object X is a square and the object X is congruent with the object Y respectively.

We will base on St. Leśniewski's mereology. The author published this system in 1916. It is presented in the revised version in [8]. The logical constants are: the equivalence symbol $\Leftrightarrow$, the negation symbol $\neg$, the implication symbol $\Rightarrow$, the disjunction symbol $\vee$, the conjunction symbol $\wedge$, the universal quantifier $\forall$ and the existential symbol $\exists$. The variables bound by a quantifier are placed directly after the sign of this quantifier.

Now we will give a short description of the system of mereology. The relation "$\leq$" is the only primitive term of mereology. The expression $X \leq Y$ is read: the object X *is a part* (proper or improper) of the object Y. The first three axioms of mereology are:

AI $\quad X \leq X.$

AII $\quad (X \leq Y \wedge Y \leq Z) \Rightarrow X \leq Z.$

AIII $\quad (X \leq Y \wedge Y \leq X) \Rightarrow X = Y.$

Thus the relation "$\leq$" is a partial order because it is reflexive (AI), transitive (AII) and half asymmetric (AIII). In order to give a shorter formulation of the last axiom of mereology two relations will be defined first.

DI $\quad X \circ Y \Leftrightarrow \exists Z\; Z \leq X, Y.$

*This work is directly connected with the paper [11].

This is read: the object X *is not disjoint* from the object Y.

Let the expression $f(X)$ mean: the object X has the property f. Then the new relation "δ" has the following form:

DII $X\delta f \Leftrightarrow \forall Z\, [f(Z) \Rightarrow Z \leq X] \wedge \forall Y\, [Y \leq X \Rightarrow \exists U\, (f(U) \wedge U \circ Y)]$

Remember that the symbol $\exists_1 X$ is read: there exists one and only one X. Now we can introduce the last axiom of mereology:

AIV $\exists Y\, [f(Y) \Rightarrow \exists_1 X\, X\delta f$.

According to AIV, for each object, which has the property f there exists one and only one object X which is in the relation δ with the property f. This object will be denoted by the symbol $\sum_A f(A)$ and will be called here the set of all those objects which have the property f or the *mereological sum*. In accordance with DII the object X is therefore the set of all the objects having the property f if and only if, every object having that property is a part of X and if for every part of X there is an object not disjoint from it, which has that property. It must be emphasised, however, that the assumption that properties exist is not necessary for constructing the foundations of mereology (see [9]). The expression $X \leq Y$ will be also read: the object X is *coverable* by the object Y or the object X *covers* the object Y. Squares will be denoted by the small letters $x, y, z, \ldots$ The relation of square congruence will be denoted by the symbol "$\equiv$". Further relations defined in this paper are denoted by the letter ϱ with an index. Let us assume now that the conjunction $x\varrho_i y \wedge x\varrho_i z$ is denoted $x\varrho_i y, z$ where ϱ_i is an arbitrary relation (analogically for the expression of larger quantity of variables). In this paper the problem of axioms is omitted (the axioms of the theory of squares are presented in [10]).

The most important notion in this paper is the notion of square cylinder, because we will define the notion of cube by this. Now we shall give some auxiliary relations before we define the notion of cube.

Definition 1
$x\varrho_1 y \Leftrightarrow \exists z\; x, y \leq z.$

Two squares are *complanar* if and only if they are a part of a square.

Definition 2
$x\varrho_2 y \Leftrightarrow \forall u \leq x\; \neg(u \leq y) \wedge \exists(b, c, d)\, [\neg(c \circ d) \wedge c, d \leq b \wedge c \leq x \wedge d \leq y \wedge b \leq \sum_a (a = x \vee a = y)]$

In conformity with Definition 2 the square x is *externally tangent* to a square y by a "side". It holds if and only if for each square u, if u is a part of the square x then u is not a part of the square y and such squares b, c, d exist that the squares c and d are disjoint and the square b is a part of the mereological sum of x and y and the squares c, d are a part of the square b and the square c is a part of x and the square d is a part of y (fig. 1).

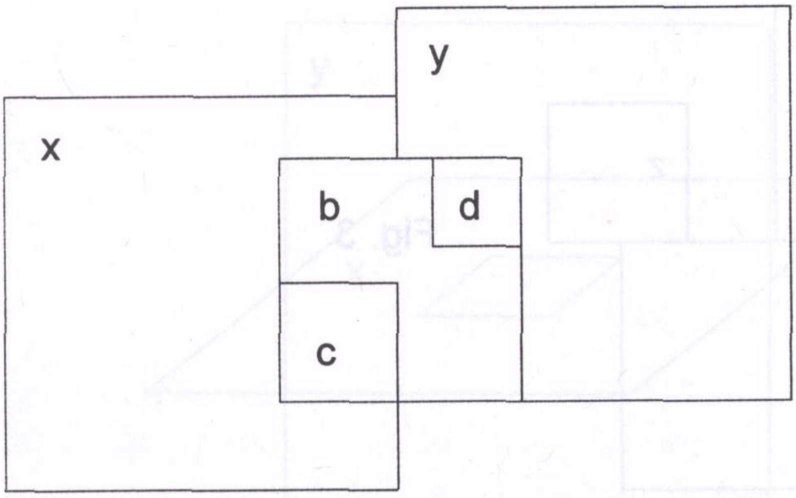

Fig. 1

Definition 3
$x\varrho_3 y \Leftrightarrow x \leq y \wedge \exists z\ z\varrho_2 x, y$

The expression $x\varrho_3 y$ is read: the square x is *internally tangent* to the square y by a "side" (fig. 2).

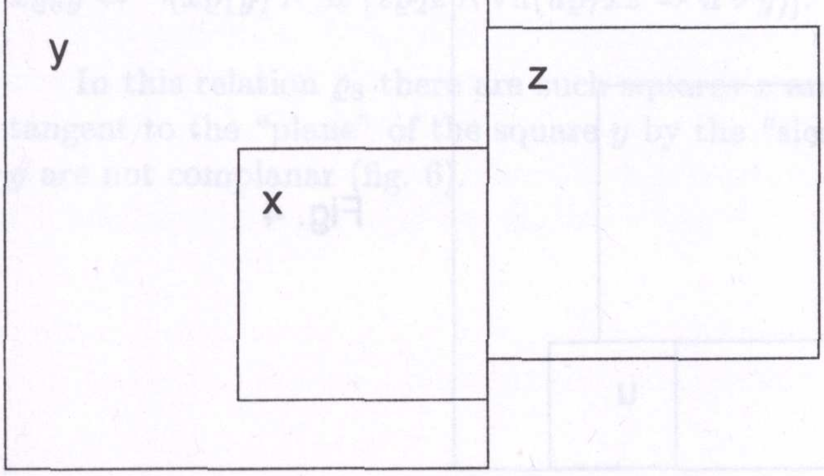

Fig. 2

Definition 4
$x\varrho_4 yz \Leftrightarrow x\varrho_2 y \wedge \forall u\varrho_3 y\,(x \leq u \Rightarrow u\varrho_2 z)$

The notation $x\varrho_4 yz$ is read: the square x is externally tangent to the squares y and z *by the same "side"* (fig. 3).

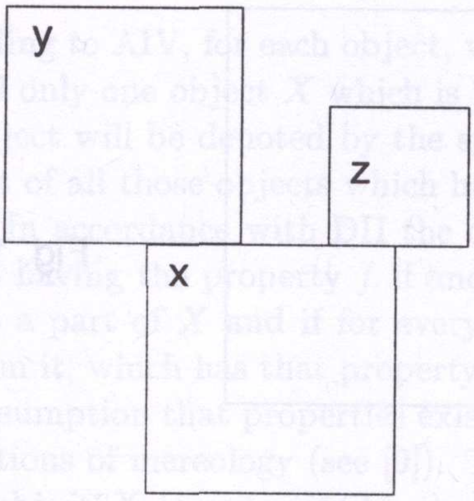

Fig. 3

Definition 5
$x\varrho_5 y \Leftrightarrow x \leq y \wedge x \neq y \wedge \exists z\, \forall u \equiv z\, (u\varrho_2 x \Rightarrow u\varrho_3 y).$

The above expression $x\varrho_5 y$ is read: the square y is *symmetrical extension* of the square x (fig. 4).

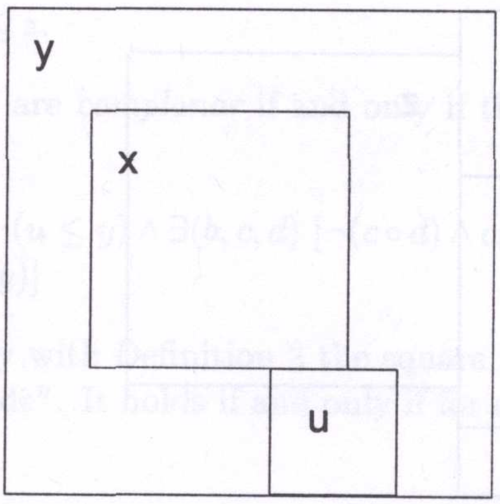

Fig. 4

Definition 6
$x\varrho_6 y \Leftrightarrow \forall z\varrho_5 x\ \forall u\varrho_5 y\ (z \circ u)$.

The relation ϱ_6 between the squares x and y takes place if and only if they are *concentric* (fig. 5).

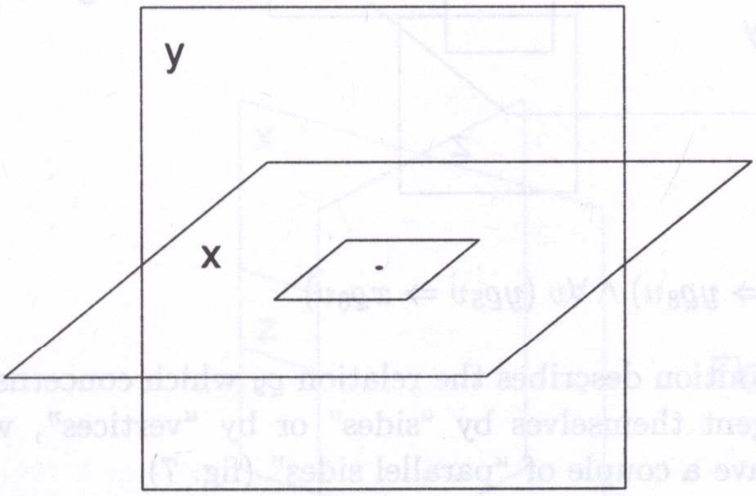

Fig. 5

Definition 7
$u\varrho_7 xy \Leftrightarrow x\varrho_2 y \wedge u \leq \sum_a (a = x \vee a = y) \wedge \exists(b, c)\ [b, c \leq u \wedge b \leq x \wedge c \leq y \wedge \neg(b \circ c)]$.

The square u is in the relation ϱ_7 with the squares x and y if and only if the squares x and y are externally tangent and square u is a part of the mereological sum of x and y and such squares b and c exist, that they are a part of u and b and c are a part of y. The square u will be called here a *connector* of x and y (fig. 1 where u is b).

Definition 8
$x\varrho_8 y \Leftrightarrow \neg(x\varrho_1 y) \wedge \exists z\ [x\varrho_2 z \wedge \forall u(u\varrho_7 xz \Rightarrow u \circ y)]$.

In this relation ϱ_8 there are such squares x and y that the square x is tangent to the "plane" of the square y by the "side" or its part and x and y are not complanar (fig. 6).

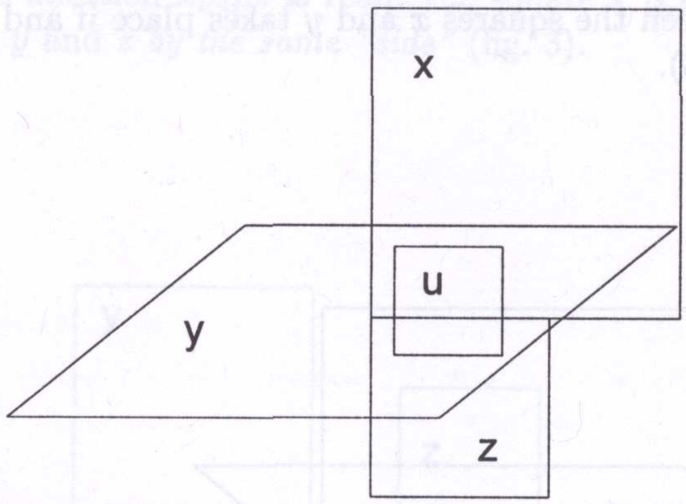

Fig. 6

Definition 9
$x\varrho_9 y \Leftrightarrow \forall u\ (x\varrho_5 u \Rightarrow y\varrho_8 u) \wedge \forall v\ (y\varrho_5 v \Rightarrow x\varrho_8 v)$

The above definition describes the relation ϱ_9 which concerns the squares x and y tangent themselves by "sides" or by "vertices", where the squares x and y have a couple of "parallel sides" (fig. 7).

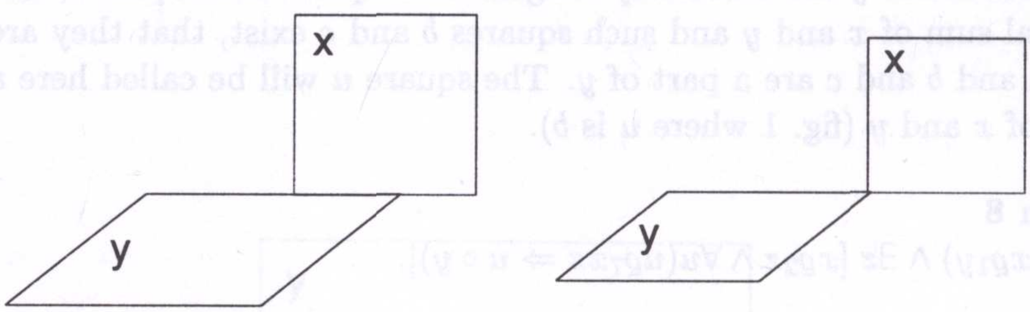

Fig. 7

Definition 10
$x\varrho_{10}y \Leftrightarrow x \neq y \wedge x\varrho_6 y \wedge x \equiv y \wedge \exists z\ z\varrho_9 x, y.$

The expression $x\varrho_{10}y$ is read: the squares x and y are *cocylindrical.* The relation between the squares x and y holds if and only if x and y are congruent and they have common "axis of symmetry". This axis of symmetry includes centres of "sides" of these squares (fig. 8).

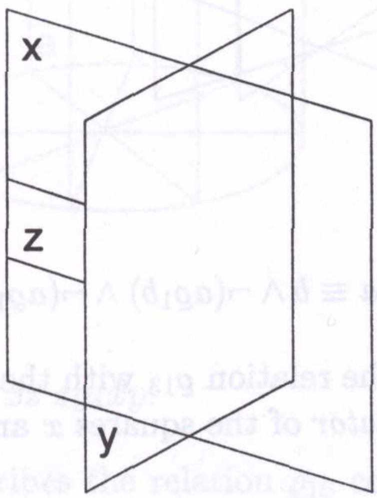

Fig. 8

Definition 11
$x\varrho_{11}y \Leftrightarrow \forall u\varrho_1 x\ \forall v\varrho_1 y\ \neg(u \circ v).$

Bi-level squares were described above. The squares x and y are *bi-level* if and only if every two squares u and v, complanar with the squares x and y respectively, are disjoint.

Definition 12
$u\varrho_{12}xy \Leftrightarrow x\varrho_{10}y \wedge u\varrho_{10}x, y \wedge \forall v\ (v\varrho_{10}x, y \Rightarrow u\varrho_{10}v).$

This definition characterizes the relation ϱ_{12} which concerns the square u *oscillating between the squares* x and y. In conformity with the Definition 12, the square u oscillates between x and y if and only if , it is cocylindrical with them and every square which is cocylindrical with the squares x and y is also cocylindrical with the square u (fig. 9).

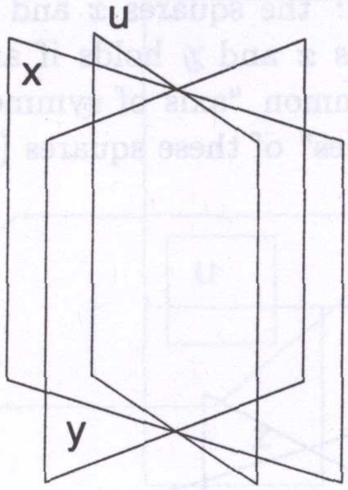

Fig. 9

Definition 13

$u\varrho_{13}xy \Leftrightarrow u\varrho_{12}xy \wedge \exists(a,b)\ [a \equiv b \wedge \neg(a\varrho_1 b) \wedge \neg(a\varrho_{11} b) \wedge a\varrho_9 u, x \wedge b\varrho_9 u, y]$.

The square u being in the relation ϱ_{13} with the squares x and y will be called here as *bisetrix oscillator* of the squares x and y.

Definition 14

$u\varrho_{14}xy \Leftrightarrow \exists(c,b)\ (c, b\varrho_{12}xy \wedge u\varrho_{13}cb)$.

The expression $u\varrho_{14}xy$ is read: the bisetrix square u oscillates between the squares x and y. It takes place if and only if there exist such squares that the square oscillates between them.

Definition 15

$Z\varrho_{15}xy \Leftrightarrow Z = \sum_a(a = x \vee a\varrho_{14}xy \vee a = y)$.

The above definition characterises the square cylinder Z determined by two congruent and cocylindrical squares x and y. In conformity with the Definition 15 the *square cylinder* Z is the mereological sum of the squares x and y and all squares oscillating between the squares x and y (fig. 10).

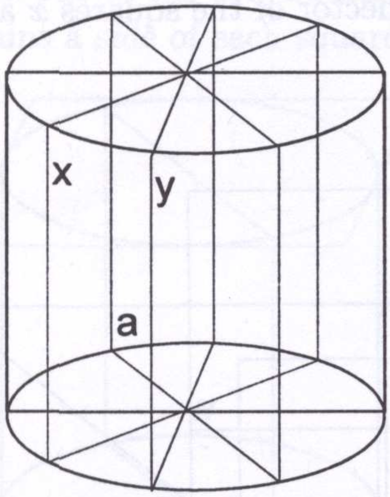

Fig. 10

Definition 16

$x\varrho_{16}y \Leftrightarrow x\varrho_2 y \wedge x \equiv y \wedge \exists z\ z\varrho_4 xy.$

This definition describes the relation ϱ_{16} concerning two twin squares x and y, which are congruent and externally tangent themselves by a side and a square externally tangent to the squares x and y by the same side exists (fig. 11).

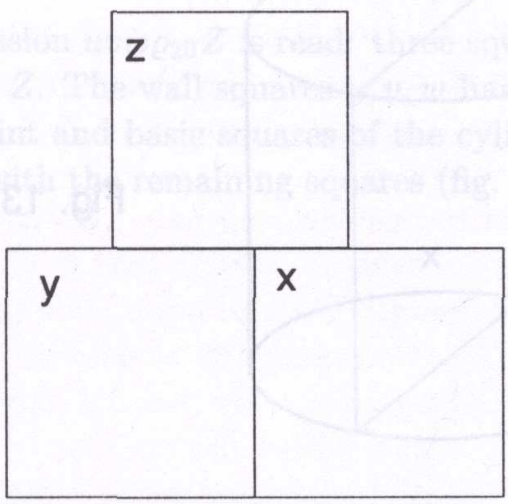

Fig. 11

Definition 17
$x\varrho_{17}y \Leftrightarrow \neg(x\varrho_1 y) \wedge \exists z\, [x\varrho_{16}z \wedge \forall u\, (u\varrho_7 xz \Rightarrow u \circ y)]$.

The expression $x\varrho_{17}y$ is read: the square y contains a side of the square x. It takes place if and only if the squares x and y are not disjoint and there is no plane containing them and there is such a square z that z and x are twin squares and every connector of the squares x and z is not disjoint with the square y (fig. 12).

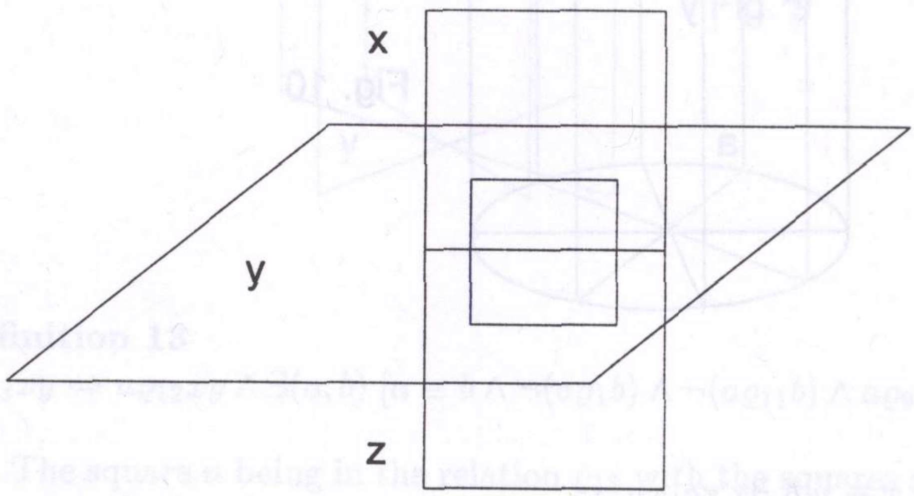

Fig. 12

Definition 18
$x\varrho_{18}Y \Leftrightarrow \exists(u,v)\, (Y\varrho_{15}uv \wedge x \equiv u) \wedge x \leq Y$.

The square x in the relation ϱ_{18} with the cylinder Y will be called here as one of the squares *forming the cylinder* Y (fig. 13).

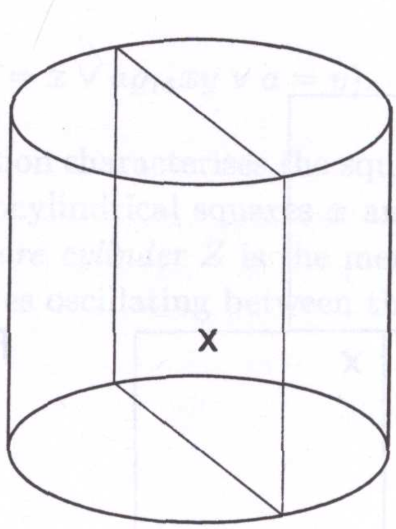

Fig. 13

Definition 19
$x\varrho_{19}Z \Leftrightarrow \exists(u,v)\ (Z\varrho_{15}uv \wedge x \equiv u) \wedge \forall w\ (w\varrho_{18}Z \Rightarrow w\varrho_{17}x).$

The square x in the relation ϱ_{19} with the cylinder Z will be called the *basic square* of the cylinder Z. This relation holds if and only if the square x is congruent with one of the cocylindrical squares of the cylinder Z and the square x contains a side of each square forming this cylinder (fig. 14).

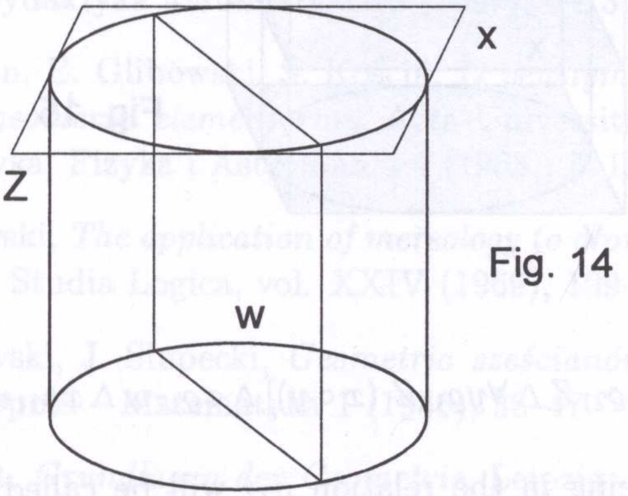

Fig. 14

Definition 20
$x\varrho_{20}y \Leftrightarrow \forall u\ (x\varrho_5 u \Rightarrow y\varrho_{17}u) \wedge \forall v\ (y\varrho_5 v \Rightarrow x\varrho_{17}v).$

The squares x and y are in the relation ϱ_{20} if and only if each square which is symmetrical extension of the square x or y contains a side of the other one. The expression $x\varrho_{20}y$ is read: the squares x and y have a *common side.*

Definition 21
$uvw\varrho_{21}Z \Leftrightarrow u, v\varrho_{19}Z \wedge \neg(u \circ v) \wedge w\varrho_{20}u, v.$

The expression $uvw\varrho_{20}Z$ is read: three squares u, v, w are wall squares of the cylinder Z. The wall squares u, v, w have such property, that two of them are disjoint and basic squares of the cylinder Z, and the third has a common side with the remaining squares (fig. 15).

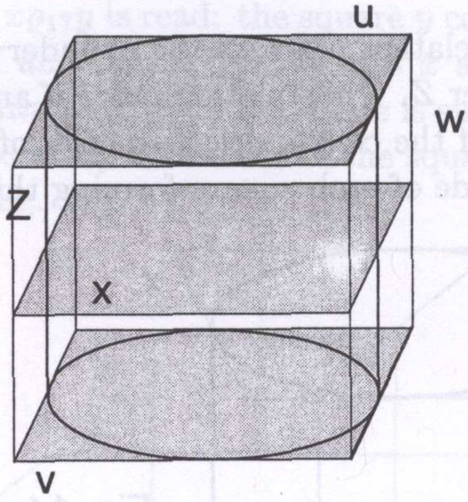

Fig. 15

Definition 22
$x\varrho_{22}uvw \Leftrightarrow \exists Z\ (uvw\varrho_{21}Z \wedge \forall y\varrho_{18}Z\ (x \circ y))] \wedge x\varrho_{17}w \wedge x\varrho_{11}u, v.$

The square x being in the relation ϱ_{22} will be called as the *drawer square* of the wall squares u, v, w of a cube. In accordance to the Definition 22 the squares x and u and x and v are bi-level and the squares u and v are basic squares of a cylinder Z and the square x has a common part with every square forming the cylinder Z. Moreover the square w contains a side of the square x (fig. 15).

Now we introduce the last definition of the notion of cube.

Definition 23
$X\varrho_{23}uvw \Leftrightarrow \exists Z\ uvw\varrho_{21}Z \wedge X = \sum_a(a = u \vee a\varrho_{22}uvw \vee a = v).$

The expression $X\varrho_{23}uvw$ is read: the object X is the *cube* determined by the squares u, v and w. It takes place if and only if there is a cylinder Z for which the squares u, v, w are wall squares and X is the mereological sum of the square u, or the square v or all drawer squares of the wall squares u, v and w.

One can see, that we can reconstruct the only primitive term of the geometry of cubes in the theory of squares. Recall, that E. Glibowski and J. Słupecki have defined intuitive counterparts of point, line and plane using only the notion of cube (see [5]). Replacing the notions of point, line and plane in Hilbert's axioms of Euclid geometry by their intuitive counterparts (i.e. the notion of sphere, the notion of infinite circular cylinder and the notion of layer respectively – all with the same given radius) we can give axioms for the geometry of cubes (the intuitive counterparts of primitive relations are easy to define).

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