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A Formalization of a Logic without Tautologies

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We are going to prove that the logic determined by a certain logical matrix is identical to the logic based on a system of rules. In our proof we use the Lindenbaum's theorem about relative maximal supersystems (see Theorem 1). The set of tautologies of the matrix under consideration is empty, this is the specyfic quality for a logic like this.

Let L be the language of sentential calculus with functors of the conjunction K, disjunction A and falsum F. We define an algebra similar to that language as follows: in the set $U = \{0, 1, ..., n\}$ we define operators k, a, f by means of the following formulas:

$k(x, y) = min(x, y)$,
$a(x, y) = max(x, y)$,
$f(x) = 0$,
for every $x, y \in U$.

Let $\mathfrak{M} = (U, \{k, a, f\}, \{1\})$ be the logical matrix for the language L. The symbol *Hom* will be used to denote the set of all homomorphisms of the language L into the matrix $\mathfrak{M}$. We define the matrix consequence $C_{\mathfrak{M}}$, determined by the matrix $\mathfrak{M}$, as follows:

Definition 1

$\alpha \in C_{\mathfrak{M}}(X) \Leftrightarrow \forall_{h \in Hom}(h(X) \subseteq \{n\} \Rightarrow h(\alpha) = n)$,
for every $\alpha \in L$ and every $X \subseteq L$.

The function $C_{\mathfrak{M}} : 2^L \to 2^L$ has the following properties:

Lemma 1

a. $X \subseteq C_{\mathfrak{M}}(X)$,

b. $X \subseteq Y \Rightarrow C_{\mathfrak{M}}(X) \subseteq C_{\mathfrak{M}}(Y)$,

c. $C_{\mathfrak{M}}(C_{\mathfrak{M}}(X)) \subseteq C_{\mathfrak{M}}(X)$, for every $X, Y \in 2^L$.

It is easy to see that $C_{\mathfrak{M}}(\emptyset) = \emptyset$, i.e., the matrix consequence $C_{\mathfrak{M}}$ has no tautologies.

The works [1 - 7] contain many results concerning logical calculi without tautologies. We use here some results of [1]. The autor uses a Hilbert-style formalisation based on the Lindenbaum's strong theorem about maximal supersystems.

For the language L we take the following set of inference rules:

$$r_1 : \frac{K\alpha\beta}{\alpha}, \quad r_2 : \frac{K\alpha\beta}{\beta}, \quad r_3 : \frac{\alpha, \beta}{K\alpha\beta},$$

$$r_4 : \frac{AA\alpha\beta\gamma}{AA\beta\alpha\gamma}, \quad r_5 : \frac{AAA\alpha\beta\gamma\delta}{AA\alpha A\beta\gamma\delta}, \quad r_6 : \frac{AA\alpha A\beta\gamma\delta}{AAA\alpha\beta\gamma\delta},$$

$$r_7 : \frac{AA\alpha\alpha\gamma}{A\alpha\gamma}, \quad r_8 : \frac{\alpha}{A\alpha\beta}, \quad r_9 : \frac{A\alpha\alpha}{\alpha},$$

$$r_{10} : \frac{AF\alpha\gamma}{\gamma}, \quad r_{11} : \frac{F\alpha}{\gamma}.$$

We now define the consequence C_R (logic C_R) on the set of rules $R = \{r_1, \ldots, r_{11}\}$ as follows:

Definition 2

$\alpha \in C_R(X) \Leftrightarrow$ there is a proof of formula α based on the set X and the set R $(X \cup \{\alpha\} \subseteq L)$.

We are going to prove that two logics $C_{\mathfrak{M}}$ and C_R are identical. Here is the above mentioned Lindenbaum's theorem:

Theorem 1

$\alpha \notin C_R(X) \Rightarrow$
$\Rightarrow \exists_Y (X \subseteq Y \wedge Y \in Syst \wedge \alpha \notin Y \wedge \forall_{\beta \notin Y} (\alpha \in C_R(Y \cup \{\beta\})))$

The family of all Lindenbaum's supersets, for the formula α and the set X, which satisfy the condition $\alpha \notin C_R(X)$ will be denoted by L_X^α.

It is casy to show the following three lemmas:

Lemma 2

a. $C_R(X \cup \{K\alpha\beta\}) = C_R(X \cup \{\alpha, \beta\})$,

b. $C_R(X \cup \{\alpha\}) \cap C_R(X \cup \{\beta\}) = C_R(X \cup \{A\alpha\beta\})$.

Lemma 3
$C_R(C_{\mathfrak{M}}(X)) \subseteq C_{\mathfrak{M}}(X)$
for every $X \subseteq L$.
The set $C_{\mathfrak{M}}(X)$ is closed under all of the rules from the set R.

Lemma 4

If $Y \in L^{\alpha}_{X}$, then the function $h_Y : L \to \{0, 1, \ldots, n\}$ definined by the formula:

$$(1)\ h_Y(\alpha) = \begin{cases} 0, & if \quad \alpha \notin Y, \\ n, & if \quad \alpha \in Y, \end{cases}$$

is a homomorphism of the language L into the matrix $\mathfrak{M}$.

To prove this lemma one can apply the properties:

(2) $K\alpha\beta \in Y \Leftrightarrow (\alpha \in Y\, and\, \beta \in Y)$,

(3) $A\alpha\beta \in Y \Leftrightarrow (\alpha \in Y\ or\ \beta \in Y)$,

(4) $F\alpha \notin Y$

where $Y \in L^{\alpha}_{X}$.
Now we prove the main theorem of this paper.

Theorem 2
$C_R = C_{\mathfrak{M}}$, i.e., the matrix $\mathfrak{M}$ is strongly adequate to the logic C_R.

Proof. Inclusion $C_R(X) \subseteq C_{\mathfrak{M}}(X)$ follows directly from lemma 3. To prove $C_{\mathfrak{M}}(X) \subseteq C_R(X)$ we assume that $\alpha \notin C_R(X)$. From theorem 1 it follows that there is a set $Y_1 \in L^{\alpha}_{X}$ with the following properties:

(1.1) $X \subseteq Y_1$,

(1.2) $Y_1 \in Syst\, (i.e., C_R(Y_1) = Y_1)$,

(1.3) $\alpha \notin Y_1$,

(1.4) $\forall_{\beta \notin Y_1} (\alpha \in C_R(Y_1 \cup \{\beta\})$.

From lemma 4 it follows that the function $h_{Y_1} : L \to \{0, 1, \ldots, n\}$ given by the formula:

$$h_{Y_1}(\gamma) = \begin{cases} 0, & if \quad \gamma \notin Y_1, \\ n, & if \quad \gamma \in Y_1, \end{cases}$$

is a homomorphism.

From the inclusions $h_{Y_1}(X) \subseteq h_{Y_1}(Y_1)$ and $h_{Y_1}(Y_1) \subseteq \{n\}$ we have $h_{Y_1}(X) = \{n\}$. From (1.3) it follows that $h_{Y_1}(\alpha) = 0$. Hence we have shown that $\exists_{h \in Hom}(h(X) \subseteq \{n\} \wedge h(\alpha) \neq n)$. This by definition 1 gives $\alpha \notin C_{\mathfrak{M}}(X)$. Then we have proved that $\alpha \notin C_R(X)$ implies $\alpha \notin C_{\mathfrak{M}}(X)$.

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