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STABILITY OF THE EQUATION OF RING HOMOMORPHISMS

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Abstract. Let $\mathcal{R}$ be a unitary ring and $(\mathcal{A}, \|\cdot\|)$ stand for a Banach algebra with a unit. In connection with some stability results of R. Badora [1] and D.G. Bourgin [2] concerning the system of two Cauchy functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \tag{$*$}$$

for mappings $f : \mathcal{R} \longrightarrow \mathcal{A}$, we deal with Hyers-Ulam stability problem for a *single* equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y)\,. \tag{$**$}$$

The basic question whether or not equation $(**)$ is equivalent to the system $(*)$ has widely been examined by J. Dhombres [3] and the present author in [4] and [5].

1. Introduction

D. G. Bourgin has shown in [2] that given a surjective map f from a ring into a Banach algebra such that both additivity and multiplicativity of f are assumed merely with some (ε, δ)-exactness, i.e.

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

and

$$\|f(xy) - f(x)f(y)\| \le \delta\,,$$

then f has to be a ring homomorphism, i.e. f has to satisfy the system of two Cauchy functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(xy) = f(x)f(y) \end{cases} \tag{$*$}$$

exactly. This stability result has been then generalized by R. Badora in [1] who was applying different methods to get rid of, among others, the surjectivity assumption upon the map in question.

The functional equation

$$f(x+y) + f(xy) = f(x) + f(y) + f(x)f(y)\,, \qquad (**)$$

resulting from summing up side by side the two Cauchy equations occurring in the system $(*)$, has been studied by J. Dhombres in [3] with the chief concern of finding possibly mild conditions quaranteeing that this *single* equation establishes a ring homomorphism. Under various alternative and less restrictive assumptions this problem was later examined by the present author in [4] and [5].

Bearing these two ideas in mind a natural question arises whether or not the functional equation $(**)$ is stable in the sense just described. In what follows, we are answering that question in affirmative.

2. The result

It should be emphasized that the so called *hyperstability* result obtained by D. G. Bourgin in [2] ((ε, δ)-exactness and the exact validity of the system are equivalent) can hardly be expected when dealing with equation $(*)$. Actually, given a positive ε a straightforward verification proves that an *arbitrary* map f from a ring into a normed algebra, enjoying the property that

$$\|f(x)\| \le \eta \qquad \text{where} \quad 4\eta + \eta^2 \le \varepsilon\,,$$

satisies equation $(**)$ with ε-exactness. Observe also that taking arbitrary elements a and r from the domain and the range of the solution f of equation $(**)$, respectively, we can easily check that the map

$$x \longmapsto af(rx)$$

yields a solution to $(**)$ as well, provided that $a^2 = a$ and $r^2 = r$. Therefore, the maps for which such shifts are bounded are, in a sense, uninteresting in the context spoken of. What about the others? The following result provides an answer to that question.

Theorem. *Let $\mathcal{R}$ be a ring with a unit 1 and let $(\mathcal{A}, \|\cdot\|)$ stand for a commutative Banach algebra with a unit e. Given an $\varepsilon \ge 0$ assume that a map $f : \mathcal{R} \longrightarrow \mathcal{A}$ is such that $f(0) = 0$, $f(1) = e$, $f(2) = 2e$, and*

$$\|f(x+y) + f(xy) - f(x) - f(y) - f(x)f(y)\| \le \varepsilon \quad \textit{for all} \quad x, y \in \mathcal{R}\,. \quad (1)$$

Then either there exist an $a \in \mathcal{A} \setminus \{0\}$ *and an* $r \in \mathcal{R} \setminus \{0\}$ *such that the map*

$$\mathcal{R} \ni x \longmapsto af(rx) \in \mathcal{A} \quad \textit{is bounded} \tag{b}$$

or

$$f \textit{ establishes a ring homomorphism between } \mathcal{R} \textit{ and } \mathcal{A}. \tag{h}$$

Proof. By setting $y = 1$ in (1) we get

$$\|f(x+1) - f(x) - e\| \le \varepsilon, \quad x \in \mathcal{R},$$

whence

$$\|f(x+2)-f(x)-2e\| \le \|f((x+1)+1)-f(x+1)-e\|+\|f(x+1)-f(x)-e\| \le 2\varepsilon,$$

holds true for all $x \in \mathcal{R}$. Now, putting $y = 2$ in (1), we infer that

$$\begin{aligned} \varepsilon &\ge \|f(x+2) + f(2x) - f(x) - 2e - 2f(x)\| \\ &= \|(f(2x) - 2f(x)) - (2e + f(x) - f(x+2))\| \\ &\ge \|f(2x) - 2f(x)\| - \|f(x+2) - f(x) - 2e\| \ge \|f(2x) - 2f(x)\| - 2\varepsilon \end{aligned}$$

and, therefore,

$$\|f(2x) - 2f(x)\| \le 3\varepsilon, \quad x \in \mathcal{R}. \tag{2}$$

A standard procedure applied already by D. H. Hyers in [6] gives now, by virtue of the completness of the algebra $(\mathcal{A}, \|\cdot\|)$, the convergence of the Hyers function sequence $(g_n)_n \in \mathbb{N}$ given by the formula

$$g_n(x) := \frac{1}{2^n} f\left(2^n x\right), \quad x \in \mathcal{R},\ n \in \mathbb{N},$$

along with the estimation

$$\|g_n(x) - f(x)\| \le 3\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right)\varepsilon, \quad x \in \mathcal{R},\ n \in \mathbb{N}, \tag{3}$$

Consequently, by setting

$$g(x) := \lim_{n\to\infty} g_n(x), \quad x \in \mathcal{R},$$

and applying (1) with x replaced by $2^n x$ we arrive at

$$\left\|\frac{1}{2^n} f\left(2^n x + y\right) + g_n(xy) - g_n(x) - \frac{1}{2^n} f(y) - g_n(x) f(y)\right\| \le \frac{1}{2^n}\varepsilon,$$

whence, passing to the limit as $n \longrightarrow \infty$, we deduce that

$$\lim_{n\to\infty} \frac{f\left(2^n x + y\right)}{2^n} = g(x)f(y) + g(x) - g(xy), \quad x, y \in \mathcal{R}. \tag{4}$$

On the other hand, with x and y replaced by $2^n x$ and $2^n y$, respectively, inequality (1) implies that

$$\left\| \frac{1}{2^n} \left[g_n(x+y) - g_n(x) - g_n(y) \right] + g_{2n}(xy) - g_n(x) \cdot g_n(y) \right\| \le \frac{1}{2^{2n}} \varepsilon \,,$$

which, after passing to the limit as $n \longrightarrow \infty$, gives the estimation

$$\| 0 \cdot [g(x+y) - g(x) - g(y)] + g(xy) - g(x) \cdot g(y) \| \le 0 \,,$$

valid for all $x, y \in \mathcal{R}$. This states that g is multiplicative, i.e.

$$g(xy) = g(x) g(y) \,, \quad x, y \in \mathcal{R} \,. \tag{5}$$

Now, equalities (4) and (5) imply that

$$\lim_{n \to \infty} \frac{f\,(2^n x + y)}{2^n} = g(x)[f(y) - g(y) + e] \,, \quad x, y \in \mathcal{R} \,,$$

whereas (3) leads obviously to

$$\| f(x) - g(x) \| \le 3\varepsilon \,, \quad x \in \mathcal{R} \,. \tag{6}$$

On setting $h := f - g + e$ we get finally that

$$\lim_{n \to \infty} \frac{f\,(2^n x + y)}{2^n} = g(x) h(y) \,, \quad x, y \in \mathcal{R} \,. \tag{7}$$

On account of the associativity of the addition in the ring $\mathcal{R}$ we derive from (1) the following two inequalities

$$\| f(x+y+z) + f((x+y)z) - f(x+y) - f(z) - f(x+y) f(z) \| \le \varepsilon$$

and

$$\| -f(x+y+z) - f(x(y+z)) + f(x) + f(y+z) + f(x) f(y+z) \| \le \varepsilon$$

valid for all triples (x, y, z) from $\mathcal{R}^3$. Summing them side by side and applying the triangle inequality we obtain the estimation

$$\begin{aligned} \| & f((x+y)z) - f(x+y) - f(z) - f(x+y) f(z) \\ & - f(x(y+z)) + f(x) + f(y+z) + f(x) f(y+z) \| \le \varepsilon \,, \end{aligned}$$

for any $(x, y, z) \in \mathcal{R}^3$.

Replacing here the variable z by $2^n z$ and dividing both sides by 2^n we arrive at

$$\|g_n((x+y)z) - \frac{1}{2^n}f(x+y) - g_n(z) - f(x+y)g_n(z)$$
$$-\frac{1}{2^n}f(2^n xz + xy) + \frac{1}{2^n}f(x) + \frac{1}{2^n}f(2^n z + y) + \frac{1}{2^n}f(x)f(2^n z + y)\| \leq \frac{1}{2^n}\varepsilon\,,$$
$$(x,y,z) \in \mathcal{R}^3\,,\ n \in \mathbb{N}.$$

Passing to the limit as $n \longrightarrow \infty$ and applying (7) we deduce that

$$g((x+y)z) - g(z) - f(x+y)g(z) - g(xz)h(xy) + g(z)h(y) + f(x)g(z)h(y) = 0\,,$$

which in view of the multiplicativity of g (see (5)) and the commutativity of $\mathcal{A}$ states that

$$[g(x+y) - e - f(x+y) - g(x)h(xy) + h(y) + f(x)h(y)]\,g(z) = 0\,, \qquad (8)$$

for every triple $(x,y,z) \in \mathcal{R}^3$.

Let $c := g(1) - e$. If $c \neq 0$, then for every $x \in \mathcal{R}$ one has $g(x) = g(1 \cdot x) = g(1)g(x)$, i.e. $c\,g(x) \equiv 0$ whence, by means of (6),

$$\|cf(x)\| = \|cf(x) - c\,g(x)\| \leq \|c\| \cdot \|f(x) - g(x)\| \leq 3\,\varepsilon\,\|c\|$$

for all $x \in \mathcal{R}$, i.e. we have (b) with $a := c$ and $r := 1$.

If $c = 0$, i.e. $g(1) = e$, then setting $z = 1$ in (8) we obtain an equation

$$h(y) - g(x)h(xy) + f(x)h(y) = f(x+y) - g(x+y) + e = h(x+y)\,, \quad x,y \in \mathcal{R}\,.$$

Since $f = h + g - e$ the latter equation may equivalently be written in the form

$$h(x+y) - h(x)h(y) = g(x)[h(y) - h(xy)]\,, \quad x,y \in \mathcal{R}\,.$$

In particular, on account of the symmetry of the left hand side, one has

$$g(x)[h(y) - h(xy)] = g(y)[h(x) - h(yx)]\,, \quad x,y \in \mathcal{R}\,,$$

whence, by setting here $y = 1$ we conclude that

$$g(x)[e - h(x)] = 0\,, \quad x \in \mathcal{R}\,,$$

because of the equality $h(1) = f(1) - g(1) + e = e$. Consequently, for all $x,y \in \mathcal{R}$ we obtain

$$g(xy)[e - h(x)] = g(y)g(x)[e - h(x)] = 0\,.$$

If we had $b := h(x_0) - e \neq 0$ for some $x_0 \in \mathcal{R} \setminus \{0\}$ (note that $h(0) = e$), we would get $bg(x_0 y) \equiv 0$ whence, by means of (6),

$$\|bf(x_0 y)\| = \|bf(x_0 y) - b\,g(x_0 y)\| \leq \|b\| \cdot \|f(x_0 y) - g(x_0 y)\| \leq 3\,\varepsilon\,\|b\|$$

for all $y \in \mathcal{R}$, i.e. we have (b) with $a := b$ and $r := x_0$.

Thus, the final possibility is: $h(x) \equiv e$ which says nothing but the equality $f = g$. Since g is multiplicative inequality (1) states that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad \text{for all} \quad x, y \in \mathcal{R}\,.$$

The celebrated D. H. Hyers theorem from [6] gives now the existence of an additive map $A : \mathcal{R} \longrightarrow \mathcal{A}$ such that

$$\|f(x) - A(x)\| \leq \varepsilon \quad \text{for every } x \in \mathcal{R}\,. \tag{9}$$

Observe now that for any $x \in \mathcal{R}$ one has

$$f(2x) = g(2x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(2^{n+1}x\right) = 2\lim_{n\to\infty} \frac{1}{2^{n+1}} f\left(2^{n+1}x\right) = 2g(x) = 2f(x)\,,$$

whence $f(2^n x) = 2^n f(x)$ for all $x \in \mathcal{R}$ and $n \in \mathbb{N}$. This jointly with (9) implies that

$$2^n \|f(x) - A(x)\| = \|f(2^n x) - A(2^n x)\| \leq \varepsilon\,, \quad x \in \mathcal{R},\ n \in \mathbb{N}\,,$$

which forces f to coincide with A. Consequently, f is both additive and multiplicative, i.e. f establishes a ring homomorphism between $\mathcal{R}$ and $\mathcal{A}$. Thus the proof has been completed.

3. Concluding remarks

The assumptions $f(0) = 0$ and $f(1) = e$ seem to be natural while dealing with homomorphisms. Note that none of them results from inequality (1). The same applies to $f(2) = 2e$; inequality (1) forces only the distance $\|f(2) - 2e\|$ to be majorized by ε. The question whether the commutativity of the target algebra is essential remains open.

The assertion of the Theorem would certainly be more readable if we had simply the alternative: either f is bounded or f is a homomorphism (classical *superstability* effect). Plainly, that is actually the case whenever both the domain ring $\mathcal{R}$ and the Banach algebra $\mathcal{A}$ in question are fields. If $\mathcal{A}$ is a field then f yields a homomorphism provided that no function of the form $x \longmapsto f(rx)$, $r \in \mathcal{R} \setminus \{0\}$, is bounded. If $\mathcal{R}$ is a field then f yields a homomorphism provided that no function af, $a \in \mathcal{A} \setminus \{0\}$, is bounded.

If either $\mathcal{R}$ or $\mathcal{A}$ has no unit the situation becomes sophisticated even while examining equation $(**)$ itself (see [4] and [5]). Thereby, the study of its stability behaviour seems to be even more difficult.

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