Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
Matematyka VI, Częstochowa 1999

Natural deduction system for some three-valued propositional logic

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The logic considered in the present paper was introduced by K.Hałkowska and A.Zając in [1]. In that work the motivations for the logic in question were given and some three-valued matrix $\mathfrak{M}$ was introduced. We shall briefly describe the mentioned motivations. Let $\mathfrak{S} =< S, \vee, \wedge, \neg >$ be a propositional language determined by an infinite denumerable set At of propositional variables and by the propositional connectives $\vee, \wedge$ and $\neg$. Furthermore, let $\mathcal{K} = \{\mathbf{-1}, \mathbf{0}, \mathbf{1}\}$, where $\mathbf{1}$ stands for true, $\mathbf{0}$ stands for false and $\mathbf{-1}$ stands for undefined. The algebra of the matrix $\mathfrak{M}$ arises from the following interpretation of disjunction, conjunction and negation:

$\alpha \vee \beta$	is defined	iff	α is defined or β is defined
$\alpha \vee \beta$	is true	iff	$\alpha$is true orβ is true
$\alpha \wedge \beta$	is defined	iff	α is defined andβ is defined
$\alpha \wedge \beta$	is true	iff	α is true andβ is true
$\neg\alpha$	is defined	iff	α is defined
$\neg\alpha$	is true	iff	α is false

Therefore, the operators $f_\vee$, $f_\wedge$ and $f_\neg$ on K are defined as follows:

$f_\wedge$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$
$\mathbf{0}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$

$f_\vee$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

	$f_\neg$
$\mathbf{-1}$	$\mathbf{-1}$
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$

As the set K^* of distinguished elements of the matrix $\mathfrak{M}$ we take the set $\{-1, 1\}$. The algebra $\mathcal{K} =< K, f_\vee, f_\wedge, f_\neg >$ is similar to to the free algebra of formulae $\mathfrak{S}$; thus, the couple $\mathfrak{M} =< \mathfrak{S}, K^* >$ is a logical matrix for the language $\mathfrak{S}$. Let us note that the set of all tautologies of the matrix $\mathfrak{M}$ is a proper subset of the set of all tautologies of classical propositional logic.

The first axiomatization of the considered logic, namely Gentzen type one, was given in [3], and a Hilbert type axiom system was given in [4].

Here we present a natural deduction formalization for which we can prove the completness theorem.

Before presenting our sequent calculus NZ let us recall some notions and introduce some notations. Let $\mathfrak{S} = < S, \vee, \wedge, \neg >$ be a propositional language. The propositional formulae (formulae, for short) will be denoted by letters $\alpha, \beta, \gamma, \delta$. The arbitrary sets of formulae will be denoted by letters X, Y, Z. By a *sequent* we mean an ordered pair $< X, \alpha >$, where $X \subseteq S$ and $\alpha \in S$. We shall write $X \vdash \alpha$ instead of $< X, \alpha >$ and $X, \alpha_1, \ldots, \alpha_n \vdash \beta$ instead of $X \cup \{\alpha_1, \ldots, \alpha_n\} \vdash \beta$.

We say that a *valuation* v (i.e. mapping from *At* into $\mathcal{K}$) *satisfies* formula a α iff $h^v(\alpha) \in K^*$, and that v *satifies* a sequent $X \vdash \alpha$ iff v satisfies α or there exists $\beta \in X$, such that $h^v(\beta) = 0$. A sequent Γ is said to be *tautological* iff every valuation satisfies Γ.

Now we define the system NZ.

Axioms of the system NZ are sequents of the form $\{\alpha\} \vdash \alpha$, where $\alpha \in S$.

Rules of inference of the system NZ are the following:

$$(r1)\ \frac{X \vdash \alpha \quad X \vdash \beta}{X \vdash \alpha \vee \beta} \qquad (r2)\ \frac{X \vdash \neg\alpha \quad X \vdash \neg\beta}{X \vdash \neg(\alpha \vee \beta)}$$

$$(r3)\ \frac{X \vdash \alpha \vee \beta \quad X \vdash \neg\alpha}{X \vdash \beta} \qquad (r4)\ \frac{X \vdash \alpha \vee \beta \quad X \vdash \neg\beta}{X \vdash \alpha}$$

$$(r5)\ \frac{X \vdash \neg(\alpha \vee \beta}{X \vdash \neg\alpha} \qquad (r6)\ \frac{X \vdash \neg(\alpha \vee \beta}{X \vdash \neg\beta}$$

$$(r7)\ \frac{X \vdash \neg\alpha}{X \vdash \neg(\alpha \wedge \beta} \qquad (r8)\ \frac{X \vdash \neg\beta}{X \vdash \neg(\alpha \wedge \beta)}$$

$$(r9)\ \frac{X \vdash \alpha \quad X \vdash \neg\alpha}{X \vdash \alpha \wedge \beta} \qquad (r10)\ \frac{X \vdash \beta \quad X \vdash \neg\beta}{X \vdash \alpha \wedge \beta}$$

$$(r11)\ \frac{X, \alpha \vdash \gamma \quad X, \beta \vdash \gamma \quad X \vdash \alpha \wedge \beta}{X \vdash \gamma}$$

$$(r12)\ \frac{X, \neg\alpha \vdash \gamma \quad X, \beta \vdash \gamma \quad X \vdash \alpha \wedge \beta}{X \vdash \gamma}$$

$$(r13)\ \frac{X, \alpha \vdash \gamma \quad X, \neg\beta \vdash \gamma \quad X \vdash \alpha \wedge \beta}{X \vdash \gamma}$$

$$(r14)\ \frac{X, \neg\alpha \vdash \gamma \quad X, \neg\beta \vdash \gamma \quad X \vdash \neg(\alpha \wedge \beta)}{X \vdash \gamma}$$

$$(r15)\ \frac{X \vdash \neg\neg\alpha}{X \vdash \alpha} \qquad (r16)\ \frac{X \vdash \alpha}{X \vdash \neg\neg\alpha}$$

$$(r17)\ \frac{X, \alpha \vdash \gamma X, \neg\alpha \vdash \gamma}{X \vdash \gamma} \qquad (r18)\ \frac{X \vdash \alpha}{X \cup Y \vdash \alpha}$$

The notion of proof is the usual one, i.e. a *proof* $\mathcal{P}$ in NZ from sequents $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a finite tree of sequents such that:

(i) every topmost sequent of $\mathcal{P}$ is either an axiom or is one of sequents $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, and

(ii) every sequent in $\mathcal{P}$ exept the lowest one is an upper sequent (*premiss*) of an inference rule whose lower sequent (*conclusion*) is also in $\mathcal{P}$.

A sequent Γ is said to be *provable in NZ from* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ if there exists a proof in NZ from $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ whose the lowest sequent is Γ. Note that the list $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ may be empty and in that case we say that Γ is *provable in NZ*.

First let us note the useful fact about the system NZ:

THEOREM 1. *For any set $X \subseteq S$ and any formulae $\gamma, \delta \in S$, if sequents $X, \delta \vdash \gamma$ and $X \vdash \delta$ are provable in NZ, then the sequent $X \vdash \gamma$ is provable in NZ.*

Since all axioms of NZ are tautological sequents and all rules of NZ are *normal* (i.e. every valuation which satisfies all premisses of a given rule does not fail to satisfy its conclusion) we can state:

THEOREM 2. *If a sequent is provable in NZ, then it is tautological.*

In order to prove the converse of the above theorem we shall adopt for the system NZ the well known Lindenbaum-Asser theorem:

THEOREM 3. *If a given sequent $X \vdash \gamma$ is not provable in NZ, then there exists a set $Y \subseteq S$, such that:*

(a) $Y \supseteq X$.

(b) $Y \vdash \gamma$ is not provable in NZ.

(c) for every $\alpha \notin Y$, the sequent $Y, \alpha \vdash \gamma$ is provable in NZ.

(d) for every $\alpha \in S, Y \vdash \alpha$ is provable in NZ iff $\alpha \in Y$.

(e) for every $\alpha \in S, \alpha \in Y$ or $\neg\alpha \in Y$.

Now we can state and prove the completness theorem for the system NZ:

THEOREM 4. *Every tautological sequent is provable in NZ.*

PROOF. Let us assume that the sequent $X \vdash \gamma$ is not provable in the system NZ. So, by the Lindenbaum-Asser theorem there exists a set Y, which satisfies conditions (a)...(e). Now we can define three sets of formulae: Y_{-1}, Y_0, Y_1 in the following way:

$$Y_0 = S - Y, \quad Y_1 = \{\alpha \in Y : \neg\alpha \in Y_0\}, \quad Y_{-1} = Y - Y_1$$

Let us observe that $Y_{-1} \subseteq Y, Y_1 \subseteq Y, Y_{-1} \cup Y_1 = Y, Y_{-1} \cup Y_0 \cup Y_1 = S$ and the sets Y_{-1}, Y_0, Y_1 are pairwise disjoint.
Moreover, using axioms and rules of the system NZ one can prove that for any formulae α and β the following conditions hold:

(i) $\neg\alpha \in Y_0 \Leftrightarrow \alpha \in Y_1$

(ii) $\neg\alpha \in Y_1 \Leftrightarrow \alpha \in Y_0$

(iii) $\neg\alpha \in Y_{-1} \Leftrightarrow \alpha \in Y_{-1}$

(iv) $\alpha \vee \beta \in Y_{-1} \Leftrightarrow \alpha \in Y_{-1}$ and $\beta \in Y_{-1}$

(v) $\alpha \vee \beta \in Y_1 \Leftrightarrow \alpha \in Y_1$ or $\beta \in Y_1$

(vi) $\alpha \vee \beta \in Y_0 \Leftrightarrow (\alpha \in Y_0$ and $\beta \notin Y_1)$ or $(\beta \in Y_0$ and $\alpha \notin Y_1)$

(vii) $\alpha \wedge \beta \in Y_1 \Leftrightarrow \alpha \in Y_1$ and $\beta \in Y_1$

(viii) $\alpha \wedge \beta \in Y_{-1} \Leftrightarrow \alpha \in Y_{-1}$ or $\beta \in Y_{-1}$

(ix) $\alpha \wedge \beta \in Y_0 \Leftrightarrow (\alpha \in Y_0$ and $\beta \notin Y_{-1})$ or $(\beta \in Y_0$ and $\alpha \notin Y_{-1})$

Now we define the valuation $v : At \to \mathcal{K}$ in the following way:

$$\text{for any } p \in At, \qquad v(p) = \begin{cases} 1, & \text{iff } p \in Y_1 \\ 0, & \text{iff } p \in Y_0 \\ -1, & \text{iff } p \in Y_{-1} \end{cases}$$

For any formula δ one can prove the following conditions:

$$\begin{aligned} h^v(\delta) &= 1 \Leftrightarrow \delta \in Y_1 \\ h^v(\delta) &= 0 \Leftrightarrow \delta \in Y_0 \\ h^v(\delta) &= -1 \Leftrightarrow \delta \in Y_{-1} \end{aligned}$$

We omit the easy proof by the induction on the length of formula δ. Eventually, let us note that the valuation h satisfies every formula from

X, as $X \subseteq Y = Y_{-1}$. Moreover $h^v(\gamma) = 0$, since $\gamma \notin Y$. This means that the valuation v does not satisfy the sequent X, so the sequent X is not a tautological sequent.

References

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