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Locally defined operators

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Let $\mathcal{F}(X,Y)$ denote the class of all functions $f: X \to Y$, where X and Y are arbitrary sets.

We start with the following

Definition. Let (X,d) be a metric space, Y, Z arbitrary sets and let $\mathcal{G} \subset \mathcal{F}(X,Y), \mathcal{H} \subset \mathcal{F}(X,Z)$ be fixed classes of functions. An operator $K: \mathcal{G} \to \mathcal{H}$ is said to be *locally defined* if and only if for every point $x_o \in X$, every $r > 0$ and for every two functions $\phi, \psi \in \mathcal{G}$ the following implication holds

$$\phi_{|B(x_o,r)} = \psi_{|B(x_o,r)} \Rightarrow K(\phi)_{|B(x_o,r)} = K(\psi)_{|B(x_o,r)}. \tag{1}$$

(The symbol $B(x_o,r)$ denotes the open ball in the metric space (X,d) centered at $x_o \in X$ and with the radious $r > 0$.)

It is easy to check the following

Remark. Let (X,d) be matric space, Y, Z arbitrary sets and let $\mathcal{G} \subset \mathcal{F}(X,Y), \mathcal{H} \subset \mathcal{F}(X,Z)$ be fixed classes of functions. If an operator $K: \mathcal{G} \to \mathcal{H}$ is locally defined then it fulfils the following, more general, condition: for every $x_o \in X, r > 0$ and $\phi, \psi \in \mathcal{G}$ the relation

$$\phi_{|B(x_o,r)} = \psi_{|B(x_o,r)} \tag{2}$$

implies the existence $z_o \in B(x_o,r)$ and $\delta > 0$ such that $B(z_o,\delta) \subset B(x_o,r)$ and

$$K(\phi)_{|B(z_o,\delta)} = K(\psi)_{|B(z_o,\delta)}. \tag{3}$$

The following example shows that the converse implication in the above Remark is not true.

Example. Let $X = [0,1], Y = Z = \mathbb{R}$ and let $x_o \in (0,1)$. Let us define an operator $K : \mathcal{F}([0,1],\mathbb{R}) \to \mathcal{F}([0,1],\mathbb{R})$ by the formula

$$K(\phi)(x) := \begin{cases} \phi(x), & x \neq x_o \\ \phi(0), & x = x_o \end{cases}, x \in [0,1].$$

Let (X,d) be a metric space and let $C^o(X,\mathbb{R})$ be the class of continuous functions $f : X \to \mathbb{R}$. In this case we have

Lemma. Let (X,d) be a metric space, Y, Z arbitrary sets and let $\mathcal{G} \subset \mathcal{F}(X,Y), \mathcal{H} \subset \mathcal{F}(X,Z)$ be fixed classes of functions. Then an operator $K : \mathcal{G} \to C^o(X,\mathbb{R})$ is locally defined if and only if for every $x_o \in X, r > 0$ and $\phi, \psi \in \mathcal{G}$ the relation (2) implies the existence $z_o \in B(x_o,r)$ and $\delta > 0$ such that $B(z_o,\delta) \subset B(x_o,r)$ and the condition (3) is fulfiled.

Proof. If an operator $K : \mathcal{G} \to C^o(X,\mathbb{R})$ is locally defined then, by Remark, it is obvious. Let us suppose that the operator K is not locally defined. Then there exist $\phi, \psi \in \mathcal{G}$, a point $x_o \in X$ and $r > 0$ such that $\phi_{|B(x_o,r)} = \psi_{|B(x_o,r)}$ and $K(\phi)_{B(x_o,r)} \neq K(\psi)_{|B(x_o,r)}$. Therefore $K(\phi)(z) \neq K(\psi)(z)$ for the certain $z \in B(x_o,r)$. The continuity of $K(\phi)$ and $K(\psi)$ implies the existence of $\delta > 0$ such that $B(z_o,\delta) \subset B(x_o,r)$ and $K(\phi)(x) \neq K(\psi)(x)$ for every $x \in B(x_o,\delta)$. So, we showed the existence $\phi, \psi \in \mathcal{G}, z \in X, \delta > 0$ such that $\phi_{|B(z,\delta)} = \psi_{|B(z,\delta)}$ and $K(\phi)_{|B(z_o,\delta_1)} \neq K(\psi)_{|B(z_o,\delta_1)}$ for every $z_o \in B(z,\delta)$ and $\delta_1 > 0$ such that $B(z_o,\delta_1) \subset B(z,\delta)$, which completes the proof.

Let $D \subset \mathbb{R}^n$ be a closed interval. By $C^m(D)$ we denote the set of all real functions defined on D such that all partial derivatives of order m exist and are continuous and by $C^\infty(D) := \bigcap_{m=0}^{\infty} C^m(D)$.

Let us notice that the balls which appear in the definition of locally defined operators can be replaced by the balls which are determined by the equivalent metrics. Therefore, from Definition and Lemma, we have the following

Corollary. Let $n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$ and let $D \subset \mathbb{R}^n$ be a closed interval. An operator $K : C^m(D) \to C^o(D)$ is locally defined if and only if for every open interval $J \subset D$ and for every two functions $\phi, \psi \in C^m(D)$ such that $\phi_{|J} = \psi_{|J}$ there exists an open interval $J^* \subset J$ such that $K(\phi)_{|J^*} = K(\psi)_{|J^*}$.
An analogous fact is also true for an operator $K : C^\infty(D) \to C^o(D)$.

Main result

During the 23-th ISFE (1985), in a personal discussion with J. Matkowski, F. Neuman asked the following question:
Does every locally defined operator $K : C^m(I) \to C^o(I)$, where I is an interval on a rational, have to be of the form

$$K(\phi)(x) = h(x, \phi(x), \phi'(x), \ldots, \phi^{(m)}(x)), \phi \in C^m(I), (x \in I),$$

for a certain function $h : I \times \mathbb{R}^{m+1} \to \mathbb{R}$?

In the paper [1] the authors gave a positive answer to this question.

Applying the Whitney Extension Theorem [2], we generalized this result. Namely, we have proved the following two theorems:

Our basic results read as follows:

Theorem 1. *Let $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and let $D \subset \mathbb{R}^n$ be a closed interval. If $K : C^m(D) \to C^o(D)$ is locally defined then there exists the unique function*

$$h : D \times \mathbb{R}^s \to \mathbb{R}, \qquad s := \sum_{s=0}^{m} \binom{n+s-1}{s},$$

such that

$$K(\phi)(x) = h(x, \phi(x), \frac{\partial \phi}{\partial x_1}(x), \ldots, \frac{\partial \phi}{\partial x_n}(x), \ldots, \frac{\partial^m \phi}{\partial x_1^m}(x), \ldots, \frac{\partial^m \phi}{\partial x_n^m}(x)),$$

for every $\phi \in C^m(D)$, $(x = (x_1, \ldots, x_n) \in D)$.

Theorem 2. *Let $D \subset \mathbb{R}^n$ be a compact interval. If $K : C^\infty(D) \to C^o(D)$ is locally defined then there exists the unique function*

$$h : D \times \mathbb{R}^N \to \mathbb{R}$$

such that for every $\phi \in C^\infty(D)$,

$$K(\phi)(x) = h(x, \phi(x), \frac{\partial \phi}{\partial x_1}(x), \ldots, \frac{\partial \phi}{\partial x_n}(x), \ldots), \quad x = (x_1, \ldots, x_n) \in D.$$

Remark. More general results, for differentiable functions in the sense of Whitney, will be published in a joint paper with J. Matkowski.

References

[1] K. Lichawski, J. Matkowski, J. Miś, *Locally defined operators in the space of differentiable functions*, Bulletin of the Polish Academy of Sciences Mathematics, Vol. 37, No. 1-6, (1989), 315-325.

[2] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36 (1934), 63-89.

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